Partial regularity of mathematical models of fluid mechanics

Wojciech Sławomir Ożański

A thesis submitted in partial fulfilment
of the requirements for the degree of
Doctor of Philosophy in Mathematics

University of Warwick, Mathematics Institute
July 2018

*“Matematyka jest najpiękniejszym
i najpotężniejszym tworem ducha ludzkiego”*

“Mathematics is the most beautiful
and the most powerful creation of the human spirit”

Stefan Banach (1938)

Acknowledgements

I would like to thank my supervisor, James Robinson, for his insightful advice, fruitful collaboration during my time as a PhD student as well as his tireless support in all matters related to the thesis and beyond.

I would also like to thank my girlfriend, Jessica, for her support and for being my inspiration.

This publication was supported by EPSRC as part of the MASDOC CDT at the University of Warwick, Grant No. EP/H023364/1.

Declaration

This thesis is submitted to the University of Warwick in support of my application for the degree of Doctor of Philosophy. It has been composed by myself and has not been submitted in any previous application for any degree.

Parts of this thesis have been submitted for publication by the author:

- Chapters 2 and 3 constitute the content of Ożański (2017),
- Chapter 4 constitute a part of Ożański & Robinson (2017),
- Chapter 5 is the subject of Ożański (2018).

List of publications including submitted papers.

- Ożański, W. S. (2017), 'On weak solutions to the Navier–Stokes inequality with internal singularities'. Submitted; preprint available at arXiv:1709.00602,
- Ożański, W. S. & Robinson, J. C. (2017), 'Partial regularity for a surface growth model'. Submitted; preprint available at arXiv:1710.07625,
- Ożański, W. S. (2018), 'A sufficient integral condition for local regularity of solutions to the surface growth model'. Submitted; preprint is available at arXiv:1803.08913.

Wojciech S. Ożański

Contents

Chapter 1

Introduction

The subject of this thesis is a mathematical study of possible singularities occurring in solutions to some equations of fluid mechanics. We will focus on the three-dimensional incompressible Navier–Stokes equations, as well as a one-dimensional surface growth model, $u_t + u_{xxxx} + \partial_{xx} u_x^2 = 0$, which appears to mimic remarkably well many mathematical features of the Navier–Stokes equations.

1.1 The Navier–Stokes equations

The Navier–Stokes equations,

$$\begin{aligned} \partial_t u - \nu \Delta u + (u\cdot \nabla) u + \nabla p = 0,\\ \operatorname{div} u = 0, \end{aligned} \tag{1.1}$$

where u denotes the velocity of a fluid, p the scalar pressure and $\nu > 0$ the viscosity, comprise a fundamental model for viscous, incompressible flows. This system of equations is supplemented with an initial condition $u(0) = u_0$, where u_0 is a given divergence-free vector field.

The fundamental mathematical theory of the Navier–Stokes equations goes back to the pioneering work of Leray (1934) (see Ożański & Pooley (2017) for a comprehensive review of Leray's work in more modern language), who showed existence and uniqueness of local-in-time *strong solutions*, namely a function u that satisfies the Navier–Stokes equations in a weak sense and such that both u and ∇u are continuous[1] on a time interval $[0,T)$ with values in the Banach space $L^2(\mathbb{R}^3)$, for some $T>0$. Moreover,

[1]As a matter of fact, Leray considered *regular motions*, which, apart from these conditions, are also required to satisfy the Navier–Stokes equations in the classical sense on time interval $(0,T)$

Leray (1934) and Hopf (1951) established global-in-time existence (without uniqueness) of weak solutions (often called *Leray-Hopf weak solutions*), which can be thought of as weak continuations of the strong solutions beyond their maximal time of existence (see below for a precise definition), in the case of the whole space $\mathbb{R}^3$ (Leray) as well as in the case of a bounded, smooth domain $\Omega \subset \mathbb{R}^3$ (Hopf).

Unless stated otherwise, we will focus on the case of the whole space $\mathbb{R}^3$.

The study of the mathematical theory of the Navier–Stokes equations is relevant since, despite significant technological progress in recent times, we are still unable to answer the fundamental question of well-posedness of the Navier–Stokes equations, that is the question of existence of finite-time blow-ups (also known as the *Navier–Stokes regularity problem*). Answering this question would have profound consequences not only in fluid mechanics, but also in numerous problems of atmospheric phenomena, physics and engineering which are based on models that include viscous flows.

In fact, its importance was acknowledged by the Clay Mathematics Institute, which, at the turn of the century, announced it as one of seven Millennium Problems.

In this work we will be interested in studying, in a sense, the "size" of the putative singular set, which we explain in some more detail in the next section.

1.2 The dimension of the set of singular times

Consider the system (1.1) with a regular initial condition u_0 (say $u_0 \in L^2(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ such that $\nabla u_0 \in L^2(\mathbb{R}^3)$) that is divergence-free. Then, as Leray (1934) proved, there exists a unique local-in-time strong solution starting from u_0, and it remains strong as long as $\|\nabla u(t)\|$ remains bounded (see, for example, Corollary 1.25 in the review Ożański & Pooley (2017) for a proof). Here $\| \cdot \|$ stands for the $L^2(\mathbb{R}^3)$ norm. He also proved that the maximal existence time T of such solution can be bounded below:

$$T \geq c \|\nabla u_0\|^{-4}, \tag{1.2}$$

(rather than in a weak sense), that certain pointwise continuity holds at $t = 0$ and moreover that u is continuous on time interval $(0, T)$ with values in $L^\infty(\mathbb{R}^3)$ and that $\|u(t)\|_{L^\infty(\mathbb{R}^3)}$ remains bounded as $t \to 0^+$, see pp. 220-221 in Leray (1934). Although such a definition can be simplified (see, for example, Definition 6.20 in Ożański & Pooley (2017)), Leray's arguments (in showing local existence and uniqueness of *regular motions*) are not based on the L^2 estimates of ∇u, but rather on L^∞ and L^2 estimates of u, see Section 6.3 in Ożański & Pooley (2017) for a wider discussion of this issue. In any case, the notion of strong solutions used in this thesis includes Leray's *regular motions*.

where c is a constant. Such a lower bound is a simple consequence of the estimates for $\|\nabla u(t)\|$. This behaviour is easy to see from the following argument, which (for simplicity) we present in the case of the torus (that is, instead of $u(t)$ being defined on the whole space $\mathbb{R}^3$ we consider $u(t)$ being defined on the three-dimensional torus $\mathbb{T}^3$, which means that $u(t)$ is periodic in all three spatial dimensions with period 2π and has mean zero for each t).

Multiplying the Navier–Stokes equation (1.1) by Δu and integrating in space, we obtain

$$\begin{aligned}\frac{1}{2}\partial_t\|\nabla u\|^2+\nu\|\Delta u\|^2 &= \int_{\mathbb{R}^3}[(u\cdot\nabla)u]\cdot\Delta u\\ &\leq \|u\|_\infty\|\nabla u\|\|\Delta u\|\\ &\leq C\|u\|_{H^1}^{1/2}\|u\|_{H^2}^{1/2}\|\nabla u\|\|\Delta u\|\\ &\leq C\|\nabla u\|^{3/2}\|\Delta u\|^{3/2}\\ &\leq \frac{\nu}{2}\|\Delta u\|^2+C\|\nabla u\|^6\end{aligned}$$

where we used Agmon's inequality $\|u\|_\infty\leq C\|u\|_{H^1}^{1/2}\|u\|_{H^2}^{1/2}$ (see Exercise 1.10 in Robinson et al. (2016) for a proof), the Poincaré inequality as well as the inequality $\|u\|_{H^2}\leq C\|\Delta u\|$ (which is easy to prove on the torus). Note also that the term involving the pressure, $\int_{\mathbb{R}^3}\nabla p\cdot\Delta u$, vanishes since u (and so also Δu) is divergence-free. Thus we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u(t)\|^2\leq C\|\nabla u(t)\|^6. \tag{1.3}$$

Therefore $\|\nabla u(t)\|^2\leq X(t)$ for $t\in[0,T')$, where $X(t)$ is the solution of the initial value problem

$$X'(t)=CX(t)^3 \quad \text{for } t\in[0,T'), \qquad X(0)=\|\nabla u(0)\|^2$$

for some $T'>0$. An easy exercise shows that

$$X(t)=\frac{\|\nabla u(0)\|^2}{\sqrt{1-2Ct\|\nabla u(0)\|^4}}, \qquad \text{for } t\in[0,T'),$$

where $T':=\frac{1}{2C}\|\nabla u(0)\|^{-4}$ is the maximal time of existence of $X(t)$, that is $X(t)\to\infty$ as $t\to T'^-$. Thus letting $T:=T'/2$ we see that

$$\|\nabla u(t)\|^2\leq X(t) \qquad \text{for } t\in[0,T],$$

and hence u remains strong until at least $T=c\|\nabla u(0)\|^{-4}$, as required.

One of the consequences of the bound (1.2) is a lower bound on $\|\nabla u(t)\|$ as t approaches the putative singular time T (if T is finite):

$$\|\nabla u(t)\| \geq C(T-t)^{-1/4}, \qquad t \in (0,T). \tag{1.4}$$

This behaviour of the L^2 norm of the gradient is useful in studying singularities in weak solutions of the Navier–Stokes equations.

A *Leray weak solution* u of the Navier–Stokes equations satisfying the initial condition u_0 is a divergence-free vector field that satisfies the Navier–Stokes equations in a weak sense, that is

$$\int_0^t \int_{\mathbb{R}^3} (u \cdot (\partial_t \phi + \nu \Delta \phi) - \phi \cdot ((u \cdot \nabla) u)) + \int_{\mathbb{R}^3} u_0 \cdot \phi(0) = \int_{\mathbb{R}^3} u(t) \cdot \phi(t) \tag{1.5}$$

for all $t > 0$ and all divergence-free test functions $\phi \in C_0^\infty(\mathbb{R}^3 \times [0,\infty); \mathbb{R}^3)$ and that also satisfies the *strong energy inequality*,

$$\|u(t)\|^2 + 2\nu \int_s^t \|\nabla u(\tau)\|^2 \mathrm{d}\tau \leq \|u(s)\|^2 \tag{1.6}$$

for all $s \in [0,\infty) \setminus \mathcal{T}$ and all $t \geq s$ for some set $\mathcal{T} \subset (0,\infty)$ of measure zero. Note that (1.6) gives in particular that

$$\int_0^\infty \|\nabla u(t)\|^2 \mathrm{d}t < \infty. \tag{1.7}$$

Such weak solutions were first constructed by Leray (1934). One motivation for studying weak solutions is the fact that they coincide with the strong solution as long as the latter exists (see, for example, Lemma 1.39 in the review Ożański & Pooley, 2017), and can therefore be thought of as a weak continuation of the strong solution. Although it is not known whether Leray weak solutions are unique[2], Leray provided a structure theorem for the set of singular times. Namely, for the weak solutions u constructed in his way the set $\mathcal{T}$ is in fact the set of putative singular times of u, that is

$$\mathcal{T} = \{t > 0 \colon \|\nabla u(t)\| = \infty\},$$

[2]However, it is remarkable that recently Buckmaster & Vicol (2017) proved that weak solutions of the Navier-Stokes equations are not unique. Here "weak solutions" is a notion of solutions that do not necessarily satisfy the energy inequality (1.6); in particular it is weaker than Leray-Hopf weak solutions and it is not clear whether such a notion can be considered a weak continuation of strong solutions.

and the set of regular times $(0,\infty)\setminus\mathcal{T}$ can be represented as

$$(0,\infty)\setminus\mathcal{T}=\bigcup_i (a_i,b_i), \tag{1.8}$$

where each (a_i,b_i) is a maximal interval of regularity; that is u coincides with some strong solution on each (a_i,b_i) and does not coincide with any strong solution on any time interval strictly containing (a_i,b_i), and the intervals (a_i,b_i) are pairwise disjoint. Note that the union is at most countable (as any union of disjoint intervals is). In particular, (1.4) gives

$$\|\nabla u(t)\|\geq c(b_i-t)^{-1/4} \qquad \text{for } t\in(a_i,b_i) \tag{1.9}$$

for every i for which b_i is finite. Using only (1.9) and (1.7) Leray estimated the size of the set of singular times $\mathcal{T}$ by concluding that

$$\sum_{i\colon b_i<\infty}\sqrt{b_i-a_i}<\infty. \tag{1.10}$$

The above bound is closely related to the *dimension* of $\mathcal{T}$. Scheffer (1976*b*) deduced from (1.10) that[3] $d_H(\mathcal{T})\leq 1/2$, where d_H denotes the *Hausdorff dimension*. Later Robinson & Sadowski (2007) deduced from (1.9) and (1.7) that

$$d_B(\mathcal{T})\leq 1/2, \tag{1.11}$$

which is also proved in Ożański & Pooley (2017; Corollary 1.43), where d_B stands for the *Minkowski dimension* (also called the *upper box-counting dimension* or the *fractal dimension*), that is

$$d_B(\mathcal{T}):=\limsup_{\varepsilon\to 0}\frac{\log N(\mathcal{T},\varepsilon)}{-\log\varepsilon}, \tag{1.12}$$

where $N(\mathcal{T},\varepsilon)$ is the minimal number of ε-balls required to cover $\mathcal{T}$. Note that $d_H(K)\leq d_B(K)$ for any compact set K, which is a fact from dimension theory (see, for example, Proposition 3.4 in Falconer, 2014). We discuss the two notions of dimension in some more detail below.

Going back to (1.10), we emphasize that the only assumptions that go into the proof of the estimates of the Hausdorff dimension d_H and the Minkowski dimension d_B of the set of singular times $\mathcal{T}$ are the growth rate of $\|\nabla u(t)\|$, as t approaches a singular

[3] In fact Scheffer (1976*b*) showed the slightly stronger property that $\mathcal{H}^{1/2}(\mathcal{T})=0$, where $\mathcal{H}^{1/2}$ denotes the 1/2-dimensional Hausdorff measure.

time (that is (1.9)) and the global integrability property (1.7). This is a central theme, which will show up repeatedly throughout the thesis.

Moreover, given conditions (1.9) and (1.7), or more generally (1.3) and (1.7), one could ask whether the resulting estimate (1.11) is the best possible. We now show that this is the case. Namely, given $T > 0$ and $\mathcal{T}' \subset (0,T)$ with $d_B(\mathcal{T}') \in (0,1/2)$ we construct a function with a singular set $\mathcal{T}'$ satisfying (1.3) and (1.7). That is we give an example of $f\colon [0,T] \to [0,\infty]$ such that

$$\int_0^T f < \infty, \qquad f'(t) \le C f(t)^3 \text{ for a.e. } t \tag{1.13}$$

and $\mathcal{T}' = \{t\colon f(t) = \infty\}$. In fact, if one is only concerned with the requirement that $d_B(\mathcal{T}') \in (0,1/2)$, then such a function can easily be constructed by letting $\alpha \in (0,1/2)$ and

$$\mathcal{T}' := \left\{ a_n \colon a_n = 1 - \frac{1}{n^{(1-\alpha)/\alpha}} : n = 1,2,\ldots \right\}.$$

Then $d_B(\mathcal{T}') = \alpha$ (which can be verified directly, see Exercise 8.1 in Robinson et al. (2016), for example) and the function f defined almost everywhere by

$$f(t) := (a_n - t)^{-\gamma} \qquad \text{for } t \in (a_{n-1}, a_n),$$

where $\gamma \in [1/2, 1 - d_B(S))$ is chosen arbitrarily, satisfies (1.13), which can be verified directly. In particular the second condition in (1.13) can be verified as follows,

$$f'(t) = \gamma (a_n - t)^{-\gamma-1} = \gamma (a_n - t)^{2\gamma-1} f(t)^3 \le \gamma T^{2\gamma-1} f(t)^3, \tag{1.14}$$

where we used the fact that $\gamma \ge 1/2$ in the last step.

Suppose now that we would like to construct a similar example for a given $\mathcal{T}'$. For instance, one could be interested in sets $\mathcal{T}'$ that have a given Hausdorff dimension (for example the uniform Cantor sets, which have the same Hausdorff dimension and the Minkowski dimension, see Example 4.5 in (Falconer 2014) for a proof of this property). In that case the simple example above becomes useless since Hausdorff dimension vanishes for any countable set. Instead we have the following lemma.

Lemma 1.1. *Let $\mathcal{T}' \subset (0,T)$ be a compact set with $d_B(\mathcal{T}') < 1/2$. Then*

$$(0,T) \setminus \mathcal{T}' = \bigcup_i (a_i, b_i), \tag{1.15}$$

where the intervals (a_i, b_i) are mutually disjoint and the union is at most countable. Moreover, let $\gamma \in [1/2, 1 - d_B(\mathcal{T}'))$ and $d : (0,T) \to \mathbb{R}$ denote the distance function *from $\mathcal{T}'$ from the left, that is*

$$d(x) := \begin{cases} b_i - x & \text{if } x \in (a_i, b_i) \text{ for some } i, \\ 0 & \text{if } x \in \mathcal{T}'. \end{cases}$$

Then the function

$$f(t) := \begin{cases} d(t)^{-\gamma} & d(t) > 0, \\ +\infty & d(t) = 0 \end{cases}$$

satisfies (1.13) *and $\mathcal{T}'$ is its singular set.*

Proof. Since $(0,T) \setminus \mathcal{T}'$ is an open set of $\mathbb{R}$ of full measure (as $d_B(\mathcal{T}') < 1/2$ implies in particular that $d_H(\mathcal{T}') < 1/2$, which in turn gives that $0 = \mathcal{H}^1(\mathcal{T}') = |\mathcal{T}'|$) the representation (1.15) follows. The second part of (1.13) follows as in (1.14). As for the first part, we note that the fact that $\gamma < 1 - d_B(\mathcal{T}')$ gives

$$\int_{\mathcal{T}'_\varepsilon} \frac{1}{\operatorname{dist}(t,\mathcal{T}')^\gamma} \mathrm{d}t < \infty \qquad \text{for every } \varepsilon > 0,$$

where $\mathcal{T}'_\varepsilon := \{ t \in \mathbb{R} : \operatorname{dist}(t,\mathcal{T}') \le \varepsilon \}$ denotes the ε-neighbourhood of $\mathcal{T}'$. This fact is a consequence of the definition of the Minkowski dimension; we refer the reader to Lemma 3.2 in Robinson & Sharples (2013) for a proof. Thus we obtain (for any $\varepsilon > 0$)

$$\begin{aligned} \int_0^T f &= \int_0^T d(t)^{-\gamma} \mathrm{d}t = \int_{[0,T]\cap \mathcal{T}'_\varepsilon} d(t)^{-\gamma}\mathrm{d}t + \int_{[0,T]\setminus \mathcal{T}'_\varepsilon} d(t)^{-\gamma}\mathrm{d}t \\ &\le \int_{[0,T]\cap \mathcal{T}'_\varepsilon} \operatorname{dist}(t,\mathcal{T}')^{-\gamma}\mathrm{d}t + \int_{[0,T]\setminus \mathcal{T}'_\varepsilon} \varepsilon^{-\gamma}\mathrm{d}t \\ &\le \int_{\mathcal{T}'_\varepsilon} \operatorname{dist}(t,\mathcal{T}')^{-\gamma}\mathrm{d}t + T\varepsilon^{-\gamma} < \infty, \end{aligned}$$

as required. □

It is important to note that $f(t)$, as described in the above lemma, has, a priori, nothing to do with the Navier–Stokes equations. In particular it is not known whether, given $\mathcal{T}'$, there exists a solution to the Navier–Stokes equations with $\|\nabla u(t)\|^2 = f(t)$ (such u would, in particular, provide a negative answer to the Navier–Stokes regularity problem!). It is not even known whether there exists a vector field $u : \mathbb{R}^3 \times [0,\infty) \to \mathbb{R}^3$ with $\|\nabla u(t)\|^2 = f(t)$, which satisfies merely the energy inequality (1.6).

Nevertheless, the above lemma models a certain type of approach which we will explore in the context of space-time singularities. Namely, the subject of this work is to study the singular set in space-time, that is the set

$$S := \{ (x,t) \in \mathbb{R}^3 \times (0,\infty) \colon u \text{ is unbounded in any neighbourhood of } (x,t) \}, \quad (1.16)$$

rather than the set of singular times. It can be shown that if $(x,t)\notin S$ then u is more regular in some neighbourhood U of (x,t). Indeed, if u is bounded on U, then $u(s)$ is smooth with respect to the space variables in U, which can be shown using the so-called local Serrin condition (see (1.32) below; see also Theorem 13.4 in Robinson et al. (2016)).

It turns out that the dimension of S can be estimated similarly to the dimension of the set of singular times $\mathcal{T}$. This is due to the partial regularity theory, which we discuss in the next section.

Remarkably, there exist "space-time" versions of the counterexample in the spirit of Lemma 1.1 and, in contrast to Lemma 1.1, they are closely related to weak solutions of the Navier–Stokes equations. Such counterexamples are the subject of a significant part of this thesis.

1.3 Partial regularity of the Navier–Stokes equations

An important contribution to the theory of the Navier–Stokes equations in the second half of the twentieth century is the partial regularity theory introduced by Scheffer (1976*a*, 1976*b*, 1977, 1978 & 1980) and subsequently developed by Caffarelli, Kohn & Nirenberg (1982). This theory is concerned with local behaviour of weak solutions, which gives rise to the notion of a "suitable" weak solution.

Definition 1.2 (Suitable weak solution of the NSE)**.** *A pair* (u,p) *is a* suitable weak solution *of the Navier–Stokes equations on* $\mathbb{R}^3\times(0,\infty)$ *if*

(i) (regularity of u and p) $u\in L^\infty((0,\infty);L^2(\mathbb{R}^3))$, $\nabla u\in L^2(\mathbb{R}^3\times(0,\infty))$, $u(t)$ is divergence-free for almost every $t\in(0,\infty)$, and $p\in L^{3/2}(\mathbb{R}^3\times(0,\infty))$,

(ii) (relation between u and p) the equation $-\Delta p=\partial_i\partial_j(u_iu_j)$ holds in the sense of distributions in $\mathbb{R}^3$ for almost every $t\in(a,b)$,

(iii) (the local energy inequality) the inequality

$$\begin{aligned}\int_{\mathbb{R}^3}|u(t)|^2\phi(t)+2\nu\int_0^t\int_{\mathbb{R}^3}|\nabla u|^2\phi\\ \le\int_0^t\int_{\mathbb{R}^3}\left(|u|^2(\partial_t\phi+\nu\Delta\phi)+(|u|^2+2p)(u\cdot\nabla)\phi\right)\end{aligned}\tag{1.17}$$

is valid for every $\phi\in C_0^\infty(\mathbb{R}^3\times(0,\infty);[0,\infty))$ and $t\in(0,\infty)$.

(iv) (the equation) the Navier–Stokes equation (1.1) *holds in the sense of distributions on* $\mathbb{R}^3 \times (0,\infty)$*, that is*

$$\int_0^\infty \int_{\mathbb{R}^3} \left(u \cdot (\partial_t \phi + \nu \Delta \phi) - \phi \cdot ((u \cdot \nabla)u) + p \operatorname{div} \phi \right) = 0 \tag{1.18}$$

for all $\phi \in C_0^\infty(\mathbb{R}^3 \times (0,\infty); \mathbb{R}^3)$.

In the thesis we will only be concerned with suitable weak solutions on $\mathbb{R}^3 \times (0,\infty)$, but one could consider instead solutions on any $U \times (a,b)$, where $U \subset \mathbb{R}^3$ is open and bounded and $a, b \in \mathbb{R}$, $a < b$. Note that the regularity assumptions on u (i.e. boundedness in time of the L^2 norm and the space-time L^2 integrability of the gradient) is the same as the regularity of Leray-Hopf weak solutions that can be deduced from the energy inequality (1.6). Using the regularity of u and the Lebesgue interpolation one obtains that $u \in L^{10/3}(\mathbb{R}^3 \times (0,\infty))$. Thus (i) implies that all terms on the right-hand side of the local energy inequality (1.17) are well defined. Moreover, if p is given by

$$p = \sum_{i,j=1}^{3} \partial_{ij} \Gamma * (u_i u_j), \tag{1.19}$$

where $\Gamma(x) \coloneqq (4\pi|x|)^{-1}$ is the fundamental solution of the Laplace equation, then (ii) is satisfied and the regularity assumption of p (i.e. $p \in L^{3/2}(\mathbb{R}^3 \times (0,\infty))$) can be deduced from the regularity of u, using the Calderón-Zygmund inequality (see, for example, Theorem B.6 in Robinson et al. (2016)). Furthermore, the distributional form of the equations (1.18) is equivalent to the weak form (1.5) (taken with t sufficiently large so that $\phi(t) = 0$), provided we restrict ourselves to the test functions $\phi \in C_0^\infty(\mathbb{R}^3 \times (0,\infty); \mathbb{R}^3)$ that are divergence-free.

An important difference between suitable weak solutions and Leray-Hopf weak solutions is that the former is a distributional solution of the NSE, while the latter is a solution to the initial value problem (i.e. (1.1) with $u(0) = u_0$). Except for this, suitable weak solutions satisfy the local energy inequality (1.17), which is an interior regularity assumption not included in the definition of Leray-Hopf weak solutions. However, given divergence-free initial data $u_0 \in L^2(\mathbb{R}^3)$, there exist Leray-Hopf weak solutions that are suitable, as was proved by Scheffer (1977) (and by Caffarelli, Kohn & Nirenberg (1982) in the case of a bounded domain). In fact, the Leray-Hopf weak solutions on $\mathbb{R}^3$ constructed by Leray (1934) are suitable, which can be deduced from Theorem 2.1 in Biryuk et al. (2007).

The central result of the partial regularity theory is the following theorem, which was proved by Caffarelli, Kohn & Nirenberg (1982).

Theorem 1.3 (Partial regularity of the Navier–Stokes equations)**.** *Let (u,p) be a suitable weak solution of the Navier–Stokes equations, and let*

$$Q_r = Q_r(z) := \{(y,s)\colon |y-x|<r, |s-t|<r^2\},$$

where $z=(x,t)$, denote a cylinder in space-time. Then

1) there exists $\varepsilon_1>0$ such that if

$$\frac{1}{r^2}\int_{Q_r}\left(|u|^3+|p|^{3/2}\right)<\varepsilon_1 \tag{1.20}$$

then u is bounded in $Q_{r/2}$;

2) there exists $\varepsilon_2>0$ such that if

$$\limsup_{r\to 0}\frac{1}{r}\int_{Q_r}|\nabla u|^2\le\varepsilon_2 \tag{1.21}$$

then u is bounded in Q_ρ for some $\rho>0$.

We note that an alternative approach to partial regularity has been developed by Lin (1998), Ladyzhenskaya & Seregin (1999), Vasseur (2007) and Kukavica (2009*b*). In fact, the results of Lin (1998) and Ladyzhenskaya & Seregin (1999) are a little different, as instead of local boundedness (as in the theorem above) they show a stronger property; namely local Hölder continuity (in space-time). In what follows we will focus on the Caffarelli et al. (1982) approach as stated above, but we will apply the approach of Lin (1998) and Ladyzhenskaya & Seregin (1999) in showing partial regularity of the *surface growth model*, which is a "mini-model" of the Navier–Stokes equations and which we discuss below.

In short, the above theorem provides sufficient conditions on the local (in space-time) behaviour of suitable weak solutions that guarantee boundedness. The partial regularity theorem is also a key ingredient in the $L_{3,\infty}$ regularity criterion for the three-dimensional Navier–Stokes equations (see Escauriaza, Seregin & Šverák 2003) and the uniqueness of Lagrangian trajectories for suitable weak solutions (Robinson & Sadowski 2009).

A remarkable feature of this partial regularity result is that the quantities involved in this result (namely $|u|^3$, $|p|^{3/2}$, $|\nabla u|^2$) are globally (in space-time) integrable for any Leray-Hopf weak solution.

Thus the above theorem implies that, given a suitable weak solution (u, p), there cannot be "too many" singular points, which is a similar observation as the one which led us to (1.11).

Corollary 1.4. *Let S denote the singular set defined by* (1.16)*. Then*

$$d_H(S) \leq 1, \qquad d_B(S \cap K) \leq 5/3 \tag{1.22}$$

for any compact $K \subset \mathbb{R}^3 \times (0, \infty)$.

Recall that d_H denotes the Hausdorff dimension and d_B denotes the Minkowski dimension. In fact, Theorem 1.3 implies a stronger estimate than $d_H(S) \leq 1$; namely that $\mathcal{P}^1(S) = 0$, where $\mathcal{P}^1(S)$ is the *parabolic Hausdorff measure* of S (see Theorem 16.2 in Robinson et al. (2016) for details).

An interesting fact about the two notions of dimension (i.e. the Hausdorff dimension and the Minkowski dimension) is that they both extend the "usual" notion of dimension to non-integer values. Namely, if $I \subset \mathbb{R}^3$ is a curve, $L \subset \mathbb{R}^3$ is a two-dimensional surface (e.g. a unit sphere) and $Q \subset \mathbb{R}^3$ is a three-dimensional body (e.g. a unit cube or ball) then

$$d_H(I) = d_B(I) = 1, \quad d_H(L) = d_B(L) = 2, \quad d_H(Q) = d_B(Q) = 3.$$

Given a compact set $K \subset \mathbb{R}^3$, $d_H(K)$ and $d_B(K)$ both measure, in a sense, its roughness. The Minkowski dimension, $d_B(K)$, is determined by counting the minimal number of r-balls required to cover K as $r \to 0^+$ (recall (1.12)), while the Hausdorff dimension, $d_H(K)$, is concerned with any covers and is calculated by summing the diameters of sets in a cover, taken to some power. To be more precise,

$$d_H(K) \coloneqq \inf\{s \geq 0 \colon \mathcal{H}^s(K) = 0\}, \tag{1.23}$$

where

$$\mathcal{H}^s(K) \coloneqq \lim_{\delta \to 0^+} \inf \left\{ \sum_k (\mathrm{diam}(B_k))^s : \mathrm{diam}(B_k) \leq \delta \text{ and } \{B_k\} \text{ covers } K \right\}.$$

It is this difference that gives the general inequality $d_H(K) \leq d_B(K)$ (which we already mentioned above) and that gives the difference in the two bounds in (1.22). Indeed, if $(x, t) \in S$ then $Q_r(x, t)$ does not satisfy (1.20) **for any** $r > 0$ while (1.21) implies merely that $\rho^{-1} \int_{Q_\rho(x,t)} |\nabla u|^2 > \varepsilon_2$ for **some** sufficiently small $\rho > 0$ (which may vary depending on (x, t)). Thus, using (1.20) we are able to construct covers of S of any given radius (and therefore deduce a bound on $d_B(S)$) while using (1.21) we are not able to guarantee the same radius of the sets in a cover (and therefore we can only deduce the bound on $d_H(S)$).

Moreover, the point of considering the intersection $S \cap K$ in (1.22) is to separate S from the set $\{(x,0)\colon x \in \mathbb{R}^3\}$. Indeed, if $(x,t) \in S$ and if $r > 0$ is given then we can make use of (1.20) only when $Q_r(x,t) \subset \mathbb{R}^3 \times (0,\infty)$. We overcome this problem by intersecting S with a compact set. This issue does not appear in the second estimate, $\mathcal{P}^1(S) = 0$.

There exist many well-known fractal sets K for which $d_H(K) = d_B(K)$ (for example Cantor sets) and also many fractal sets K for which $d_H(K) < d_B(K)$ (for example the set $K \coloneqq \{n^\alpha \colon n \in \mathbb{N}\} \subset \mathbb{R}$ satisfies $d_B(K) = (\alpha+1)^{-1}$, while $d_H(K) = 0$, since the Hausdorff dimension vanishes for countable sets (which can be seen directly from the definition (1.23))).

Therefore studying the bounds on $d_B(S \cap K)$, $d_H(S)$ for a putative singular set S of suitable weak solutions of the Navier–Stokes equations might provide a valuable information about the complexity of the putative singularities. For example, it would be interesting to know whether (at least in some cases) $d_H(S)$, $d_B(S \cap K)$ admit the same bound. Perhaps it is possible to show that (in some cases) $d_H(S) = d_B(S \cap K)$ or $d_H(S) < d_B(S \cap K)$. Moreover, it is interesting to try to improve the bounds in (1.22), which would limit (in a sense) the size of the singularities. In fact, it turns out that the bound on $d_B(S \cap K)$ can be improved: Kukavica (2009*a*) showed that $d_B(S \cap K) \le 135/82 (\approx 1.65)$. This bound was later refined by Kukavica & Pei (2012), Koh & Yang (2016) down to the most recent bound $d_B(S \cap K) \le 2400/1903 (\approx 1.261)$ obtained by He et al. (2017). As for the Hausdorff dimension, the bound $d_H(S) \le 1$ has not been improved. In fact, the ingenious construction of counterexamples by Scheffer (1985 & 1987) shows that it is sharp in the sense that the result of Theorem 1.3 cannot be improved given the ingredients used by Caffarelli et al. (1982).

To be precise, from the properties (i)-(iv) of suitable weak solutions only (i)-(iii) are used in the proof of Theorem 1.3 due to to Caffarelli, Kohn & Nirenberg (1982). In other words the distributional form of the Navier–Stokes equations is irrelevant to the claim of Theorem 1.3 (we note, however, that this is not the case to the alternative proofs of the theorem, due to Lin (1998), Ladyzhenskaya & Seregin (1999), Vasseur (2007) and Kukavica (2009*b*)). This gives rise to the notion of a weak solution to the Navier–Stokes inequality.

Definition 1.5 (Weak solutions to the Navier–Stokes inequality). *A pair* (u,p) *is a* weak solution to the Navier–Stokes inequality *on* $\mathbb{R}^3\times(0,\infty)$ *if it satisfies conditions (i)-(iii) of Definition 1.2.*

We use the name *Navier–Stokes inequality* since the local energy inequality (1.17),

$$\int_{\mathbb{R}^3}|u(t)|^2\phi(t)+2\nu\int_0^t\int_{\mathbb{R}^3}|\nabla u|^2\phi \\ \leq\int_0^t\int_{\mathbb{R}^3}\left(|u|^2(\partial_t\phi+\nu\Delta\phi)+(|u|^2+2p)(u\cdot\nabla)\phi\right),$$

is a weak form of the inequality

$$u\cdot(\partial_t u-\nu\Delta u+(u\cdot\nabla)u+\nabla p)\leq 0. \tag{1.24}$$

Indeed, assuming that (u,p) is smooth (in both space and time) we can rewrite (1.24) in the form

$$\frac{1}{2}\partial_t|u|^2-\frac{\nu}{2}\Delta|u|^2+\nu|\nabla u|^2+u\cdot\nabla\left(\frac{1}{2}|u|^2+p\right)\leq 0,$$

where we used the calculus identity $u\cdot\Delta u=\Delta(|u|^2/2)-|\nabla u|^2$. Multiplication by 2ϕ and integration by parts gives (1.17).

Furthermore, setting

$$f\coloneqq\partial_t u-\nu\Delta u+(u\cdot\nabla)u+\nabla p, \tag{1.25}$$

one can (formally) think of the Navier–Stokes inequality (1.24) as the inhomogeneous Navier–Stokes equations with forcing f,

$$\partial_t u-\nu\Delta u+(u\cdot\nabla)u+\nabla p=f,$$

where f acts against the direction of the flow u, that is $f\cdot u\leq 0$.

Since Theorem 1.3 applies to weak solutions of the Navier–Stokes inequality, one can similarly consider the singular set S defined as in (1.16), that is

$$S\coloneqq\{(x,t)\in\mathbb{R}^3\times(0,\infty)\colon u \text{ is unbounded in any neighbourhood of } (x,t)\}.$$

We note that, in contrast to suitable weak solutions of the NSE (recall the comments following (1.16)), one cannot deduce any further regularity of u at points $(x,t)\notin S$ (as the local Serrin condition (mentioned below (1.16)) does not necessarily apply to weak solutions of the Navier–Stokes inequality). As in (1.22) one can deduce the following.

Theorem 1.6 (Partial regularity of weak solutions to the NSI)**.** *Let u be a weak solution to the Navier–Stokes inequality. Then the claims of Theorem 1.3 remain true. In particular*

$$d_H(S) \leq 1,$$

where S is the singular set of u.

We can now state Scheffer's counterexample.

Theorem 1.7 (Scheffer's counterexample)**.** *Given $\xi \in (0,1)$ there exists a weak solution to the Navier–Stokes inequality with*

$$\xi \leq d_H(S) \leq 1.$$

A study of such a counterexample is a significant part of the thesis. We discuss this, together with other results in the next section.

1.4 The main results of the thesis

The results of the thesis can be separated into three parts. First, we present the counterexample of Scheffer in a simplified way. This allows for a number of remarkable insights into the structure of the Navier–Stokes equations and the Navier–Stokes inequality, and in particular we discuss in detail the role of the pressure function as the mechanism responsible for obtaining blow-up. In addition to this we refine the constructions to obtain some new results. For example, we consider the case of "almost equality",

$$-\vartheta \leq u \cdot (\partial_t u - \nu \Delta u + (u \cdot \nabla) u + \nabla p) \leq 0$$

for any preassigned $\vartheta > 0$ (rather than merely (1.24)). We show how, with such a constraint, one can still construct a blow-up on the Cantor set (with the Hausdorff dimension greater than ξ, as before). Remarkably, a simple argument shows that such an extension is possible only when $\nu = 0$ (see Theorem 2.3 and Section 3.3), which provides a new insight into the so-called *vanishing viscosity problem.* In the case of the "almost equality" with $\nu > 0$ (instead of a blow-up) we obtain a norm inflation property,

$$\| u(1) \|_{L^\infty} \geq \mathcal{N} \| u(0) \|_{L^\infty},$$

for any given $\mathcal{N}$. See Theorem 2.4 and Section 3.3 for details.

Second, we study the regularity problem for the *surface growth model*,

$$u_t + u_{xxxx} + \partial_{xx} u_x^2 = 0, \tag{1.26}$$

which is a one-dimensional fourth order partial differential equation. This equation originates from a model of molecular epitaxy; that is a physical process of constructing semiconductors via a continuous flow of particles, see King et al. (2003), Siegert & Plischke (1994) and Raible et al. (2000) for details. Remarkably, the surface growth model shares a number of striking similarities with the three-dimensional incompressible Navier–Stokes equation. These include the exact analogues of local-in-time existence and uniqueness results of strong solutions, global-in-time existence of weak solutions, as well as other topics, such as blow-up rates, weak-strong uniqueness and upper bounds on the dimension of the set of singular times $\mathcal{T}$. In fact, one can check that each result in Leray's (1934) analysis of local-in-time strong solutions to the Navier–Stokes equations has an analogue (with some minor modifications) in the case of the surface growth model (considered on the whole line $\mathbb{R}$). In the case of the torus $\mathbb{T}$ Stein & Winkler (2005) proved existence of global-in-time weak solutions, which we now define.

Definition 1.8 (Weak solutions of the SGM)**.** *Suppose that $u_0 \in L^2(\mathbb{T})$ has zero mean. We say that u is a (global-in-time)* weak solution *of the surface growth initial value problem*

$$\begin{cases} u_t = -u_{xxxx} - \partial_{xx} u_x^2 & \text{in } \mathbb{T}\times(0,\infty), \\ u(0) = u_0, \end{cases}$$

if $\int_{\mathbb{T}} u(t) = 0$ for almost every $t \geq 0$, and that for every $T > 0$

$$u \in L^\infty((0,T);L^2(\mathbb{T})) \cap L^2((0,T);H^2(\mathbb{T}))$$

and

$$-\int_0^T \int \left(u\,\phi_t - u_{xx}\phi_{xx} - u_x^2 \phi_{xx} \right) = \int u_0 \phi(0)$$

for all $\phi \in C_0^\infty(\mathbb{T}\times[0,T))$.

Later Blömker & Romito (2009) proved local existence in the critical space $\dot{H}^{1/2}$ and (spatial) smoothness for solutions bounded in $L^{8/(2\alpha-1)}((0,T);H^\alpha)$ for any $1/2 < \alpha < 9/2$, and in Blömker & Romito (2012) they proved local existence in a critical space of a similar type to that occurring in the paper by Koch & Tataru (2001) for the Navier–Stokes equations. As in the case of the Navier–Stokes equations, the question of the existence of finite-time singularities of solutions to the surface growth model is open. This (at least partially) justifies our claim that studying the regularity problem in the surface growth model can be thought of as studying the regularity problem of the Navier–Stokes equations in a simpler setting. Indeed, any progress on the regularity problem for the surface growth model could inspire a breakthrough in the case of the Navier–Stokes equations.

It turns out that a mathematical analysis of the surface growth model reveals its unique structure and gives rise to new techniques, some of which seem to be applicable in other systems (for example a nonlinear parabolic Poincaré inequality, see (1.30) below).

A partial regularity theory for the surface growth model (in the spirit of Lin (1998) and Ladyzhenskaya & Seregin (1999)) is presented in Chapter 4. Similarly as in the case of the NSE, this theory is concerned with suitable weak solutions, that is with weak solutions such that the local energy inequality

$$\frac{1}{2}\int u(t)^2\phi(t) + \int_0^t\int_{\mathbb{T}} u_{xx}^2\phi \le \int_0^t\int_{\mathbb{T}}\left(\frac{1}{2}(\phi_t-\phi_{xxxx})u^2 + 2u_x^2\phi_{xx} - \frac{5}{3}u_x^3\phi_x - u_x^2 u\phi_{xx}\right) \tag{1.27}$$

holds for all $\phi \in C_0^\infty(\mathbb{T}\times(0,T);[0,\infty))$ and all $t\ge 0$.

The main result of Chapter 4 is the following.

Theorem 1.9 (Partial regularity of the surface growth model)**.** *Let u be a suitable weak solution of the surface growth model on the torus $\mathbb{T}$, and let*

$$Q_r = Q_r(z) \coloneqq \{(y,s)\colon |y-x|<r, |s-t|<r^4\},$$

where $z=(x,t)$, denote a cylinder in space-time (observe that a space-time cylinder of this form scales differently from the cylinder defined in Theorem 1.3). Then

1) there exists $\varepsilon_1>0$ such that if

$$\frac{1}{r^2}\int_{Q_r}|u_x|^3<\varepsilon_1$$

then u is Hölder continuous in $Q_{r/2}$;

2) there exists $\varepsilon_2>0$ such that if

$$\limsup_{r\to 0}\frac{1}{r}\int_{Q_r}u_{xx}^2\le\varepsilon_2$$

then u is Hölder continuous in Q_ρ for some $\rho>0$.

As in the case of the Navier–Stokes equation (recall Corollary 1.4), this result lets us estimate the dimension of the singular set.

Corollary 1.10. *The singular set S of a weak solution to the surface growth model enjoys the estimates*

$$d_H(S) \leq 1, \qquad d_B(S \cap K) \leq 7/6 \tag{1.28}$$

for any compact $K \subset \mathbb{T} \times (0, \infty)$.

Here

$$S := \{(x,t) \in \mathbb{T} \times [0,\infty) : \ u \text{ is not space-time Hölder continuous in any neighbourhood of } (x,t)\}. \tag{1.29}$$

The above theorem and corollary is an analogue of the partial regularity theory for the Navier–Stokes equations. In fact, as far as I know, the surface growth model is the only lower-dimensional model for which this kind of analogue has been proved. Note also that the bound on $d_B(S)$ is sharper than the best known corresponding bound in the case of Navier–Stokes equations (recall (1.22)).

The proof of Theorem 1.9 is not based on the approach of Caffarelli et al. (1982), which does not seem well suited for the surface growth model. Instead our approach is inspired by the work of Ladyzhenskaya & Seregin (1999). Furthermore, a crucial new idea of our proof is a nonlinear parabolic Poincaré inequality, which states that any distributional solution to the surface growth model on a cylinder $Q_r = Q_r(x,t)$ satisfies

$$\| u - [u]_{Q_r} \|_{L^3(Q_r)} \leq C\, r \left(\| u_x \|_{L^3(Q_r)} + r^{-2/3} \| u_x \|^2_{L^3(Q_r)} \right), \tag{1.30}$$

where

$$[u]_{Q_r} := \frac{1}{|Q_r|} \int_{Q_r} u$$

denotes the mean of u over Q_r. Observe that there is no time derivative on the right-hand side of (1.30), which is the main point of this result. We discuss this inequality in Section 4.3.

Note that the definition of the singular set differs from the one in the case of the Navier–Stokes equations, see (1.16). In fact, it is not known if a locally bounded weak solution to the surface growth model is (locally) Hölder continuous. We discuss this issue in more detail in Chapter 4.

Finally, in Chapter 5 we provide a certain integral condition that guarantees local regularity of weak solutions to the surface growth model. Namely we show that if

$$u_x \in L^{q'}((t_1,t_2);L^q(U)) \quad \text{with } \frac{4}{q'}+\frac{1}{q} \le 1, \tag{1.31}$$

(we also assume that $q' < \infty$) then u is smooth in $U \times (t_1,t_2)$. Such a condition is inspired by the corresponding local regularity condition in the case of the Navier–Stokes equations: if u is a weak solution on a space-time domain $U \times (t_1,t_2)$ such that

$$u \in L^{q'}((t_1,t_2);L^q(U)) \quad \text{with } \frac{2}{q'}+\frac{3}{q} \le 1 \tag{1.32}$$

then u is smooth in the space variables on this domain. This is often called the *local Serrin condition*, and is named after Serrin (1962 & 1963), who was the first to study the property (1.32).

Remarkably, under the condition (1.31) we obtain smoothness in both space and time (rather than smoothness in space only). Moreover, we will see that it does not seem possible to gain "a derivative at a time" in the proof of local regularity. We will circumvent this problem by using fractional derivatives to gain, roughly speaking, "one half of a derivative at a time".

An interesting fact about this result is that the partial regularity theory (Theorem 1.9) immediately shows that the regularity condition (1.31) implies α-Hölder continuity for any $\alpha \in (0,1)$ if $q,q' \ge 3$ and $q' < \infty$. Indeed, Hölder's inequality gives

$$\frac{1}{r^2}\int_{Q(z,r)} |u_x|^3 \le \|u_x\|^3_{L^{q',q}(Q(z,r))} r^{3(1-1/q-4/q')},$$

which is less than ε_0 if the cylinder $Q(z,r)$ is small enough. Thus the condition for partial regularity (recall the first claim of Theorem 1.9) can be guaranteed for each sufficiently small cylinder $Q(z,r)$, and the claim follows. It is therefore interesting to observe how the condition (1.31) gives C^∞ smoothness, rather than merely α-Hölder continuity for any $\alpha \in (0,1)$. This also suggests that the definition of the singular set (1.29) is optimal.

1.5 Notation

In this section we briefly explain the notation we adopt throughout the thesis.

We will say that a function is *smooth* on an open set if it is of class C^∞ on this set. We use the notation ∂_λ for the partial derivative with respect to a variable λ. We often simplify the notation corresponding to the partial derivative with respect to x_i by writing

$$\partial_i \equiv \partial_{x_i}.$$

We do not apply the summation convention over repeated indices. We use a standard notation regarding the Lebesgue spaces L^p, and Sobolev spaces $W^{k,p}$ and $H^k := W^{k,2}$, see for example Adams & Fournier (2003). We will mostly focus on function spaces defined on the whole space $\mathbb{R}^3$ or on the 1-dimensional torus $\mathbb{T}$, which should be clear from the context.

We denote by v_ε the mollification of a function $v : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ in both variables.

Chapter 2

Weak solutions to the Navier–Stokes inequality with internal singularities

In this chapter we discuss Scheffer's counterexample mentioned in Theorem 1.7 above. We will prove the following result, first shown by Scheffer (1985).

Theorem 2.1 (Weak solution of NSI with point singularity). *There exist $\nu_0 > 0$ and a function $\mathfrak{u}\colon \mathbb{R}^3 \times [0,\infty) \to \mathbb{R}^3$ that is a weak solution of the Navier–Stokes inequality with any $\nu \in [0,\nu_0]$ such that $\mathfrak{u}(t) \in C^\infty$, $\operatorname{supp} \mathfrak{u}(t) \subset G$ for all t for some compact set $G \Subset \mathbb{R}$ (independent of t). Moreover $\mathfrak{u}$ is unbounded in every neighbourhood of (x_0, T_0), for some $x_0 \in \mathbb{R}^3$, $T_0 > 0$.*

Recall Definition 1.5 for the definition of a weak solution to the NSI.

Observe that the above theorem gives a velocity field $\mathfrak{u}$ that satisfies the Navier–Stokes inequality for all $\nu \in [0,\nu_0]$ at the same time. Moreover, using an appropriate rescaling, it is clear that the statement of the above theorem is equivalent to the one where $\nu_0 = 1$ and $(x_0, T_0) = (0,1)$. Indeed, if $\mathfrak{u}$ is the velocity field given by the theorem then $\sqrt{T_0/\nu_0}\,\mathfrak{u}(x_0 + \sqrt{T_0\nu_0}x, T_0 t)$ satisfies Theorem 2.1 with $\nu_0 = 1$, $(x_0, T_0) = (0,1)$.

In a subsequent paper Scheffer (1987) constructed weak solutions of the Navier–Stokes inequality that blow up on a Cantor set $S \times \{T_0\}$ with $d_H(S) \geq \xi$ for any preassigned $\xi \in (0,1)$.

Theorem 2.2 (Nearly one-dimensional singular set). *Given any $\xi \in (0,1)$ there exists $\nu_0 > 0$, a compact set $G \Subset \mathbb{R}^3$ and a function $\mathfrak{u}\colon \mathbb{R}^3 \times [0,\infty) \to \mathbb{R}^3$ that is a weak solution to the Navier–Stokes inequality such that $u(t) \in C^\infty$, $\operatorname{supp} \mathfrak{u}(t) \subset G$ for all t, and*

$$\xi \leq d_H(S) \leq 1,$$

where

$$S := \{(x,t) \in \mathbb{R}^3 \times (0,\infty) \, : \, \mathfrak{u}(x,t) \text{ is unbounded in any neighbourhood of } (x,t)\}.$$

We discuss the above theorem in Chapter 3. The above results make use of an alternative form of the local energy inequality. Namely, the local energy inequality (1.17) is satisfied if the *local energy inequality on the time interval* $[S,S']$,

$$\begin{aligned}\int_{\mathbb{R}^3} |u(x,S')|^2 \phi \,\mathrm{d}x - \int_{\mathbb{R}^3} |u(x,S)|^2 \phi \,\mathrm{d}x + 2\nu \int_S^{S'} \int_{\mathbb{R}^3} |\nabla u|^2 \phi \\ \leq \int_S^{S'} \int_{\mathbb{R}^3} \left(|u|^2 + 2p\right) u \cdot \nabla \phi + \int_S^{S'} \int_{\mathbb{R}^3} |u|^2 \left(\partial_t \phi + \nu \Delta \phi\right),\end{aligned} \tag{2.1}$$

holds for all $S, S' > 0$ with $S < S'$, which is clear by choosing S, S' such that $\operatorname{supp} \phi \subset \mathbb{R}^3 \times (S,S')$. An advantage of this alternative form of the local energy inequality is that it demonstrates how to combine solutions of the Navier–Stokes inequality one after another. Namely, (2.1) shows that a necessary and sufficient condition for two vector fields $u^{(1)} \colon \mathbb{R}^3 \times [t_0,t_1] \to \mathbb{R}^3$, $u^{(2)} \colon \mathbb{R}^3 \times [t_1,t_2] \to \mathbb{R}^3$ satisfying the local energy inequality on the time intervals $[t_0,t_1]$, $[t_1,t_2]$, respectively, to combine (one after another) into a vector field satisfying the local energy inequality on the time interval $[t_0,t_2]$ is that

$$|u^{(2)}(x,t_1)| \leq |u^{(1)}(x,t_1)| \qquad \text{for a.e. } x \in \mathbb{R}^3. \tag{2.2}$$

It turns out that Scheffer's dense proofs of the two theorems can be rephrased in a more succinct and intuitive form, which we present in this chapter. As a part of the simplification process we introduce the notion of a *structure* on an open subset of the upper half-plane (see Definition 2.7), which allows one to construct a compactly supported, divergence-free vector field u in $\mathbb{R}^3$ with prescribed absolute value $|u|$ and with a number of other useful properties (see Section 2.2.4 and Lemma 2.5). Moreover, we point out the key concepts used in the construction of the blow-up. Namely, we introduce the notion of the *pressure interaction function* (corresponding to a given subset of the half-plane and its structure, see Section 2.2.6), which articulates a certain nonlocal property of the pressure function (see Lemma 2.9), and we formalise the concept of the *geometric arrangement* (see Section 2.3), that is a certain configuration of subsets of the upper half-plane (and their structures) which, in a sense, "magnifies" the pressure interaction. We also expose some other concepts used in the proof, such as an analysis of rescalings of vector fields and some ideas related to dealing with the nonlocal character of the pressure function. In addition to these simplifications, we point out how Theorem 2.2 is obtained as a straightforward extension of Theorem 2.1.

Furthermore, we improve Theorem 2.2 in the case $\nu_0 = 0$ to construct weak solutions to the Navier–Stokes inequality that, except for a finite-time blow-up on a Cantor set, also satisfy the "approximate equality"

$$-\vartheta \leq u \cdot (\partial_t u + (u \cdot \nabla) u + \nabla p) \leq 0 \tag{2.3}$$

for any preassigned $\vartheta > 0$, in the sense that we now make precise. We will divide the time interval $(0, \infty)$ into countably many disjoint open intervals $\{I_k\}$ such that $\bigcup \overline{I_k} = (0, \infty)$, where $\overline{I_k}$ denotes the closure of I_k. We will be concerned with vector fields u that are smooth on $\mathbb{R}^3 \times I_k$ for each k and that $u|_{I_{k+1}}$ and $u|_{I_k}$ can be combined (one after another) as in (2.2). Such a switching procedure (which will become clear in the section below) does not allow us to define the time derivative at the switching time and for this reason we will understand (2.3) in the sense that

$$-\vartheta \leq u \cdot (\partial_t u + (u \cdot \nabla) u + \nabla p) \leq 0 \quad \text{everywhere in } \mathbb{R}^3 \times I_k \text{ for every } k. \tag{2.4}$$

The definition of the intervals $\{I_k\}$ will be a part of the construction of u.

In constructing such an improvement we use the construction from the proof of Theorem 2.2 and present a simple argument showing how the approximate equality requirement (with any ϑ) enforces $\nu = 0$; we thereby obtain the following result.

Theorem 2.3. *Given $\xi \in (0,1)$ and $\vartheta > 0$ there exists a function $\mathfrak{u} \colon \mathbb{R}^3 \times [0, \infty) \to \mathbb{R}^3$ satisfying the claim of Theorem 2.2 with $\nu = 0$ such that* (2.3) *holds with $\nu = 0$ (i.e.* (2.4) *holds with $\nu = 0$ for some choice of the intervals $\{I_k\}$).*

In other words, there exists a divergence-free solution to the inhomogeneous Euler equation,

$$\partial_t u + (u \cdot \nabla) u + \nabla p = f,$$

with the forcing f "almost orthogonal" to the velocity field, that is $-\vartheta \leq u \cdot f \leq 0$, and that blows up on the Cantor set.

It is not clear how to obtain a weak solution to the Navier–Stokes inequality (with some $\nu > 0$) that blows up and satisfies the approximate equality. However, one can sharpen Scheffer's constructions to obtain the following "norm inflation" result.

Theorem 2.4 (Smooth solution of NSI with norm inflation)**.** *Given $\mathcal{N} > 0$, $\vartheta > 0$ there exists $\eta > 0$ and a nontrivial solution $\mathfrak{u} \in C^\infty(\mathbb{R}^3 \times (-\eta, 1+\eta); \mathbb{R}^3)$ to the Navier–Stokes inequality* (1.24) *satisfying the approximate equality*

$$\| u \cdot (\partial_t u - \nu \Delta u + (u \cdot \nabla) u + \nabla p) \|_{L^\infty} \leq \vartheta, \tag{2.5}$$

for all $\nu \in [0,1]$, *supp* $\mathfrak{u}(t) = G$ *for all* t *(where* $G \subset \mathbb{R}^3$ *is compact), and*

$$\|u(1)\|_{L^\infty} \geq \mathcal{N} \|u(0)\|_{L^\infty}.$$

The two theorems above are discussed in Chapter 3. The results of this and the next chapter have been posted on the ArXiv (see Ożański (2017)) and have been submitted to form the main part of volume in the Springer subseries "Lecture Notes in Mathematical Fluid Mechanics".

The structure of this chapter is as follows. In Section 2.1 below we present a sketch of the proof of Theorem 2.1. In the following Section 2.1.1 we observe some of the basic properties of the vector field $\mathfrak{u}$ obtained in the sketch and we point out how such a vector field can be used as a benchmark for various results in the theory of the Navier–Stokes equations, particularly blow-up criteria. The sketch of the proof of Theorem 2.1 is based on the existence of certain objects, which, after introducing a number of preliminary concepts in Section 2.2, we construct in Section 2.3. The construction of these objects is based on a certain "geometric arrangement", which we discuss in Section 2.4.

2.1 Sketch of the proof of Theorem 2.1

Here we present a simple argument which proves Theorem 2.1 given the following assumptions. Namely, suppose for a moment that there exists $T > 0$, a compact set $G \subset \mathbb{R}^3$ and a divergence-free vector field u such that $u \in C^\infty(\mathbb{R}^3 \times [0,T]; \mathbb{R}^3)$, $\operatorname{supp} u(t) = G$ for all $t \in [0,T]$, and the Navier–Stokes inequality

$$\partial_t |u|^2 \leq -u \cdot \nabla \left(|u|^2 + 2p \right) + 2\nu \, u \cdot \Delta u \tag{2.6}$$

holds in $\mathbb{R}^3 \times [0,T]$ for all $\nu \in [0,\nu_0]$ for some $\nu_0 > 0$, where $p(t)$ is the pressure function corresponding to u (recall (1.19)). Here $C^\infty(\mathbb{R}^3 \times [0,T]; \mathbb{R}^3)$ is a short-hand notation for the space of vector functions that are infinitely differentiable on $\mathbb{R}^3 \times (-\eta, T+\eta)$ for some $\eta > 0$.

Suppose further that, during time interval $[0,T]$ u admits the following interior gain of magnitude property: that for some $\tau \in (0,1)$, $z \in \mathbb{R}^3$ the affine map

$$\Gamma(x) \coloneqq \tau x + z,$$

maps G into itself and that, at time T, u attains a large gain in magnitude; namely that

$$|u(\Gamma(x), T)| \geq \tau^{-1} |u(x,0)|, \qquad x \in \mathbb{R}^3. \tag{2.7}$$

Such a gain in magnitude allows us to consider a rescaled copy of u and, in a sense, slot it into the part of the support G in which the gain occurred. Namely, considering

$$u^{(1)}(x,t) := \tau^{-1} u(\Gamma^{-1}(x), \tau^{-2}(t-T))$$

we see that $u^{(1)}$ satisfies the Navier–Stokes inequality (2.6) on $\mathbb{R}^3 \times [T,(1+\tau^2)T]$, $\mathrm{supp}\, u^{(1)}(t) = \Gamma(G)$ for all $t \in [T,(1+\tau^2)T]$ and that (2.7) gives

$$\left| u^{(1)}(x,T) \right| \leq |u(x,T)| , \qquad x \in \mathbb{R}^3 \tag{2.8}$$

(and so u, $u^{(1)}$ can be combined "one after another", recall (2.2)). Thus, since $u^{(1)}$ is larger in magnitude than u (by the factor of τ) and its time of existence is $[T,(1+\tau^2)T]$, we see that by iterating such a switching we can obtain a vector field $\mathfrak{u}$ that grows indefinitely in magnitude, while its support shrinks to a point (and thus will satisfy all the claims of Theorem 2.1), see Fig. 2.1. To be more precise we let $t_0 := 0$,

$$t_j := T \sum_{k=0}^{j-1} \tau^{2k} \qquad \text{for } j \geq 1,$$

$T_0 := \lim_{j\to\infty} t_j = T/(1-\tau^2)$, $u^{(0)} := u$, and

$$u^{(j)}(x,t) := \tau^{-j} u\left(\Gamma^{-j}(x), \tau^{-2j}(t-t_j)\right), \qquad j \geq 1, \tag{2.9}$$

see Fig. 2.1. As in (2.8), (2.7) gives that

$$\mathrm{supp}\, u^{(j)}(t) = \Gamma^j(G) \qquad \text{for } t \in [t_j, t_{j+1}] \tag{2.10}$$

and that the magnitude of the consecutive vector fields shrinks at every switching time, that is

$$\left| u^{(j)}(x,t_j) \right| \leq \left| u^{(j-1)}(x,t_j) \right| , \qquad x \in \mathbb{R}^3, j \geq 1, \tag{2.11}$$

see Fig. 2.1.

Thus letting

$$\mathfrak{u}(t) := \begin{cases} u^{(j)}(t) & \text{if } t \in [t_j, t_{j+1}) \text{ for some } j \geq 0, \\ 0 & \text{if } t \geq T_0, \end{cases} \tag{2.12}$$

we obtain a vector field that satisfies the claims of Theorem 2.1. Indeed, by construction $\mathfrak{u}$ is divergence-free, smooth in space, its support in space is contained in G, and $\mathfrak{u}$ is unbounded in every neighbourhood of (x_0, T_0), where

$$\{x_0\} := \bigcap_{j\geq 0} \Gamma^j(G) = \left\{ \frac{z}{1-\tau} \right\}.$$

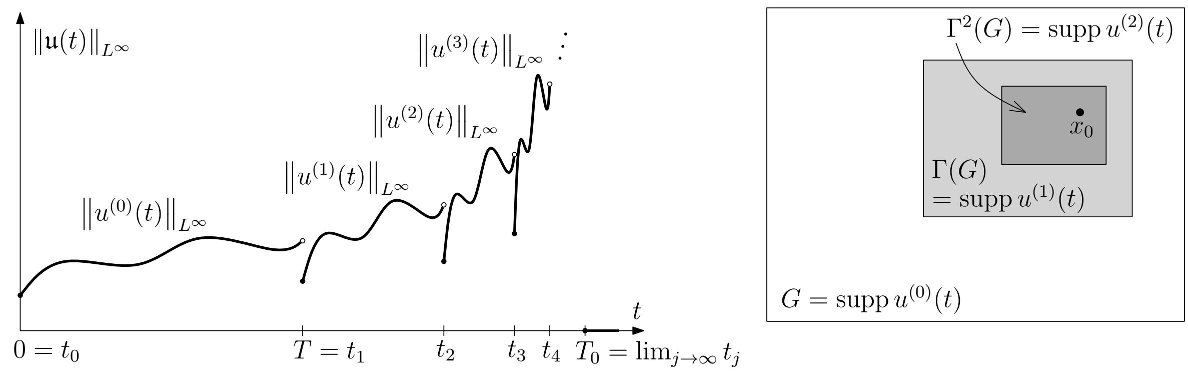

Figure 2.1: The switching procedure: the blow-up of $\|\mathfrak{u}(t)\|_\infty$ (left) and the shrinking support of $\mathfrak{u}(t)$ (right) as $t \to T_0^-$.

As for the regularity $\sup_{t>0}\|u(t)\| < \infty$ and $\nabla u \in L^2(\mathbb{R}^3 \times (0,\infty))$ (recall Definition 1.5) we write for any $t \in [t_j, t_{j+1}]$, $j \geq 0$,

$$\|\mathfrak{u}(t)\| = \|u^{(j)}(t)\| \leq \sup_{t\in[t_j,t_{j+1})} \|u^{(j)}(t)\| = \tau^{j/2} \sup_{t\in[t_0,t_1]} \|u^{(0)}(t)\| \leq \sup_{t\in[t_0,t_1]} \|u^{(0)}(t)\| < \infty, \tag{2.13}$$

where we used the fact that $\tau \in (0,1)$. Similarly,

$$\int_0^\infty \|\nabla \mathfrak{u}(t)\|^2 = \sum_{j=0}^\infty \int_{t_j}^{t_{j+1}} \|\nabla u^{(j)}(t)\|^2 = \int_{t_0}^{t_1} \|\nabla u^{(0)}(t)\|^2 \sum_{j=0}^\infty \tau^j < \infty, \tag{2.14}$$

as required.

As for the local energy inequality (1.17), we see that, by construction, the local energy inequality (2.1) is satisfied on any time interval $[S,S'] \subset [0,T_0)$. Since $\|\mathfrak{u}(t)\| \to 0$ as $t \to T_0^-$ (since $\tau^{2j} \to 0$ as $j \to \infty$, see the calculation above) and the regularity $\sup_{t>0}\|u(t)\| < \infty$, $\nabla u \in L^2(\mathbb{R}^3 \times (0,\infty))$ gives global-in-time integrability of all the terms appearing under the space-time integrals in (2.1) the Dominated Convergence Theorem lets us take the limit $S' \to T_0$ to obtain the local energy inequality on any interval $[S,S'] \subset [0,\infty)$, as requried.

Therefore we have established the proof of Theorem 2.1 given the existence of T, G, u, ν_0, τ and z with the properties listed above. These objects are constructed in Section 2.3 (which includes a particularly enlightening proof of the Navier–Stokes inequality (2.6), see Section 2.3.2). We now discuss some interesting properties of the vector field $\mathfrak{u}$ which are a consequence of the above switching procedure.

2.1.1 Remarks

Note that $\mathfrak{u}$ enjoys a self-similar property

$$\mathfrak{u}(x_0 - x, T_0 - s) = \tau^j \mathfrak{u}(x_0 - \tau^j x, T_0 - \tau^{2j} s), \qquad x \in \mathbb{R}^3, s \in (0, T_0], j \geq 0,$$

which is also the property characteristic for the Leray hypothetical self-focusing strong solutions to the Navier–Stokes equations (that is (3.12) in Leray (1934), in which $x_0 = 0$; note however such solutions do not exist, as was shown by Nečas, Růžička & Šverák (1996)), except that here the self-similarity holds only for the discrete scaling factors τ^j, $j \geq 0$.

Moreover, $\mathfrak{u}$ satisfies the energy inequality

$$\|\mathfrak{u}(\tau_2)\|_{L^2}^2 + 2\nu \int_{\tau_1}^{\tau_2} \|\nabla \mathfrak{u}(t)\|_{L^2}^2 \mathrm{d}t \leq \|\mathfrak{u}(\tau_1)\|_{L^2}^2, \qquad \nu \in [0, \nu_0] \tag{2.15}$$

for every $\tau_1 \in [0, \infty)$ such that $\tau_1 \notin \{t_j\}_{j \geq 1}$, and every $\tau_2 > \tau_1$, which can be verified as follows. Let $1 \leq j_1 \leq j_2$ and take

$$\phi(x, t) = \psi(x) \mathcal{T}(t),$$

where $\psi \in C_0^\infty(\mathbb{R}^3)$ is such that $\psi \geq 0$, $\psi = 1$ on G and $\mathcal{T} \in C_0^\infty((0, \infty))$ is such that $\mathcal{T} = 1$ on $[t_{j_1}, t_{j_2}]$ and $\operatorname{supp} \mathcal{T} \subset (t_{j_1 - 1}, t_{j_2 + 1})$. Then the local energy inequality (2.1) gives

$$2\nu \int_{t_{j_1-1}}^{t_{j_2+1}} \mathcal{T}(t) \|\nabla \mathfrak{u}(t)\|_{L^2}^2 \mathrm{d}t \leq \int_{t_{j_1-1}}^{t_{j_1}} \|\mathfrak{u}(t)\|_{L^2}^2 \mathcal{T}'(t)\, \mathrm{d}t + \int_{t_{j_2}}^{t_{j_2+1}} \|\mathfrak{u}(t)\|_{L^2}^2 \mathcal{T}'(t)\, \mathrm{d}t.$$

Given $\varepsilon > 0$ and $\tau_1 \in (t_{j_1-1}, t_{j_1})$, $\tau_2 \in (t_{j_2}, t_{j_2+1})$ let $\mathcal{T}(t) := J_\varepsilon \chi_{(\tau_1, \tau_2)}(t)$, where χ is an indicator function and J_ε denotes the (usual) mollification operator. Given such a choice of $\mathcal{T}$ we can use the smoothness of $\mathfrak{u}$ on each of the intervals (t_j, t_{j+1}), $j \geq 0$ to take the limit $\varepsilon \to 0^+$ in the inequality above to obtain the energy inequality (2.15) for $\tau_1 \in (t_{j_1-1}, t_{j_1})$, $\tau_2 \in (t_{j_2}, t_{j_2+1})$. Thus, since $\mathfrak{u}$ is right-continuous in time and its magnitude does not increase at a switching time (recall (2.11)), the last inequality is valid also for $\tau_1 \in [t_{j_1-1}, t_{j_1})$, $\tau_2 \in [t_{j_2}, t_{j_2+1}]$, as required.

Furthermore, although the vector field $\mathfrak{u}$ is not a solution of the Navier–Stokes equations, it can be used to benchmark some results in the theory of these equations, for example the regularity criteria. A regularity criterion is a condition guaranteeing that a local-in-time strong solution $u(t)$ of the Navier–Stokes equations on a time interval $[0, T)$ does not blow-up as $t \to T^-$. For example, $u(t)$ does not blow-up if it satisfies any of the following.

(1) The *Beale-Kato-Majda criterion* (due to Beale et al. (1984)):

$$\int_0^T \|\operatorname{curl} u(t)\|_\infty < \infty,$$

(2) The *Serrin condition* (due to Serrin (1963)):

$$\int_0^T \|u(t)\|_{L^s}^r < \infty \qquad \text{for any } s \geq 3, r \geq 2 \text{ satisfying } \frac{2}{r} + \frac{3}{s} = 1,$$

or

(3) *Control of the direction of vorticity* (due to Constantin & Fefferman (1993)):

$$\text{for some } \Omega, \rho > 0 \quad \left| P^{\perp}_{\xi(x,t)}(\xi(x+y,t)) \right| \leq |y|/\rho \tag{2.16}$$

for x, y, t such that

$$t \in [0,T], \quad |\operatorname{curl} u(x,t)|, |\operatorname{curl} u(x+y,t)| > \Omega.$$

Here $\xi(x,t) \coloneqq \operatorname{curl} u(x,t)/|\operatorname{curl} u(x,t)|$ is the direction of vorticity $\operatorname{curl} u(x,t)$, and $P^{\perp}_x y \coloneqq \sin\alpha$, where α denotes the angle between the vectors $x, y \in \partial B(0,1) \subset \mathbb{R}^3$.

Remarkably, $\mathfrak{u}$ does not satisfy any of the above criteria, which is a consequence of the switching argument applied in the previous section (as for (3) above note that the direction of $\operatorname{curl} u^{(0)}$ is not constant and so the direction of $u^{(j)}$ cannot be controlled as in (2.16) as $j \to \infty$).

However, $\mathfrak{u}$ does satisfy the $L_{3,\infty}$ *criterion* (due to Escauriaza et al. (2003), see also Seregin (2007, 2012)): if

$$\|u(t)\|_{L^3} \text{ remains bounded as } t \to T^-$$

then $u(t)$ (a local-in-time strong solution on time interval $[0,T)$) does not blow-up as $t \to T^-$. Indeed the L^3 norm of $\mathfrak{u}(t)$ remains bounded by $\sup_{t\in[0,T]} \|u^{(0)}(t)\|_{L^3}$. This shows that the $L_{3,\infty}$ regularity criterion uses, in an essential way, properties of solutions of the Navier–Stokes equations (rather than merely the Navier–Stokes inequality (1.24)).

We emphasize that the above property constitute an important difference between solutions to the Navier–Stokes inequality and the Navier–Stokes equations. Not only it shows that the $L_{3,\infty}$ condition does not hold in the case of the NSI, but also that the blow-up scenario (2.12) presented in the previous section cannot occur in the case of the NSE. Furthermore, we note that a local-in-time strong solution of the NSE cannot have compact support in space (which can be shown by considering the vorticity equation and applying a unique continuation theorem (see Theorem 4.1 in Escauriaza et al. (2003), for example). The same is true of Leray-Hopf weak solutions of the NSE, since any such solution is strong at times not belonging to the set of singular times $\mathcal{T}$ (recall (1.8)).

In the next three sections we complete the sketch of the proof of Theorem 2.1, that is we construct constants $T>0$, $\eta>0$, $\nu_0>0$, $\tau\in(0,1)$, $z\in\mathbb{R}^3$, the set G and the vector field u with the properties listed in the beginning of Section 2.1. For this we first introduce a number of preliminary results regarding rotationally invariant vector fields in $\mathbb{R}^3$, properties of the pressure function as well as introduce the concept of a *structure* on a subset U of the upper half plane (Section 2.2). Then, in Section 2.3, we perform the construction of $T>0$, $\eta>0$, $\nu_0>0$, $\tau\in(0,1)$, $z\in\mathbb{R}^3$, G, u and we show the required claims. The construction is based on a certain *geometric arrangement*, which is the heart of the proof of Theorem 2.1 and which we discuss in detail in Section 2.4.

2.2 Preliminaries

We will say that a function is *smooth* on an open set if it is of class C^∞ on this set. We use the notation ∂_λ for the partial derivative with respect to a variable λ. We often simplify the notation corresponding to the partial derivative with respect to x_i by writing

$$\partial_i \equiv \partial_{x_i}.$$

We do not apply the summation convention over repeated indices. We let

$$P := \{(x_1,x_2)\in\mathbb{R}^2 \colon x_2>0\}$$

denote the upper half plane. We frequently use the convention

$$h_t(\cdot)\equiv h(\cdot,t), \tag{2.17}$$

that is the subscript t denotes dependence on t (rather than the t-derivative, which we denote by ∂_t). By writing

"*outside* G" we mean "*for* $x\notin G$".

By $\overline{U}$ we denote the closure of an open set U. We often write that a function is *a solution to* a theorem (or proposition/lemma) if it satisfies the claim of the theorem.

2.2.1 The rotation R_ϕ

We denote by R_ϕ the rotation around the x_1 axis by an angle ϕ, that is

$$R_\phi(x_1, x_2, x_3) = (x_1, x_2 \cos\phi - x_3 \sin\phi, x_2 \sin\phi + x_3 \cos\phi).$$

We will refer to R_ϕ (for some ϕ) simply as *the rotation*, since it is the only operation of rotation that we consider in the thesis. It is clear that any $x \in \mathbb{R}^3$ is either a point on the x_1 axis, a point in P or a rotation $R_\phi(y_1, y_2, 0)$ of some point y of P by some angle $\phi \in (0, 2\pi)$. For $U \Subset P$ set

$$R(U) \coloneqq \{x \in \mathbb{R}^3 \,:\, x = R_\phi(y, 0) \text{ for some } \phi \in [0, 2\pi),\ y \in U\}, \tag{2.18}$$

the *rotation of* U (see Fig. 2.2). Clearly, if U_1, U_2 are disjoint subsets of P then $R(U_1)$, $R(U_2)$ are disjoint subsets of $\mathbb{R}^3$. We will denote by $R^{-1}\colon \mathbb{R}^3 \to \overline{P}$ the *cylindrical projection* defined by

$$R^{-1}(y_1, y_2, y_3) \coloneqq \left(y_1, \sqrt{y_2^2 + y_3^2} \right). \tag{2.19}$$

The projection R^{-1} is in fact the left-inverse of R, that is $R^{-1}R = \mathrm{id}$. It is not a right-inverse, but $R\,R^{-1}(V) \supset V$ for any $V \subset (\mathbb{R}^3 \setminus Ox_1)$ (where Ox_1 denotes the x_1 axis), as is clear from Fig. 2.2. We say that a velocity field $u\colon \mathbb{R}^3 \to \mathbb{R}^3$ is *rotationally invariant* (axially symmetric) if

$$u(R_\phi x) = R_\phi u(x) \qquad \text{for } \phi \in [0, 2\pi), x \in \mathbb{R}^3 \tag{2.20}$$

while a scalar function $q\colon \mathbb{R}^3 \to \mathbb{R}$ is *rotationally invariant* if

$$q(R_\phi x) = q(x) \qquad \text{for } \phi \in [0, 2\pi), x \in \mathbb{R}^3$$

in other words $q(x) = q(R^{-1}x)$ for $x \in \mathbb{R}^3$. Observe that if a velocity field $u \in C^2$ and a scalar function $q \in C^1$ are rotationally invariant then the vector function $(u \cdot \nabla)u$ and the scalar functions

$$|u|^2, \quad \operatorname{div} u, \quad u \cdot \nabla |u|^2, \quad u \cdot \nabla q, \quad u \cdot \Delta u \quad \text{and} \quad \sum_{i,j=1}^{3} \partial_i u_j \partial_j u_i \tag{2.21}$$

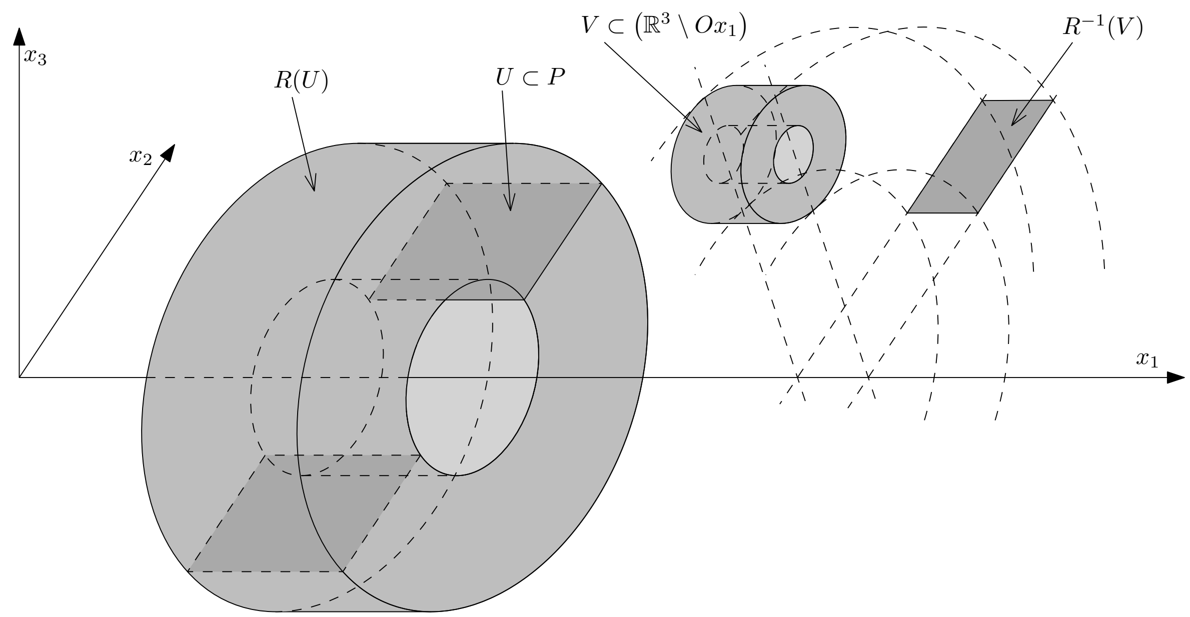

Figure 2.2: The rotation R and the cylindrical projection R^{-1}.

are rotationally invariant. These facts can be shown by a simple calculation and by making use of the algebraic identity

$$\sum_{i,j=1}^{3} \partial_i u_j \partial_j u_i = \operatorname{div}\left((u\cdot\nabla)u\right) - u\cdot\nabla(\operatorname{div} u),$$

see Appendix 2.5.2 for details.

2.2.2 The pressure function

Given a vector field $u\colon \mathbb{R}^3 \to \mathbb{R}^3$ consider the pressure function $p\colon \mathbb{R}^3 \to \mathbb{R}$ corresponding to u, that is

$$p(x) \coloneqq \int_{\mathbb{R}^3} \sum_{i,j=1}^{3} \frac{\partial_i u_j(y)\partial_j u_i(y)}{4\pi|x-y|}\,\mathrm{d}y,$$

recall (1.19). Here we briefly comment on some geometric properties of the pressure function, which will be crucial in constructing a velocity field u satisfying the Navier–Stokes inequality (2.6) (see for instance Lemma 2.9).

First, if $u \in C_0^\infty(\mathbb{R}^3)$ then the corresponding pressure function is smooth on $\mathbb{R}^3$ with

$$|\nabla p(x)| \le \widetilde{C}|x|^{-4} \quad \text{and} \quad |D^2 p(x)| \le \widetilde{C}|x|^{-5} \tag{2.22}$$

for some $\widetilde{C} > 0$ (which depends on u), which follows from integration by parts. Moreover, p satisfies the limiting property

$$\lim_{x_1 \to \pm\infty} x_1^4 \partial_1 p(x_1, 0, 0) = \frac{\pm 3}{4\pi} \int_{\mathbb{R}^3} \left(|u(y)|^2 - 3u_1^2(y) \right) \mathrm{d}y, \tag{2.23}$$

which can be verified directly by simple algebra. Finally, if $u \in C_0^\infty(\mathbb{R}^3)$ is rotationally invariant then the change of variable $z = R_{-\phi} y$ and (2.21) give

$$p(R_\phi x) = \int_{\mathbb{R}^3} \sum_{i,j=1}^3 \frac{\partial_i u_j(y) \partial_j u_i(y)}{4\pi |R_\phi x - y|} \mathrm{d}y = \int_{\mathbb{R}^3} \sum_{i,j=1}^3 \frac{\partial_i u_j(R_\phi z) \partial_j u_i(R_\phi z)}{4\pi |x - z|} \mathrm{d}z = p(x)$$

for all $\phi \in [0, 2\pi)$. That is the pressure function corresponding to a rotationally invariant vector field is rotationally invariant.

2.2.3 The functions $u[v,f]$, $p[v,f]$

Now let v be a 2D vector field and f be a scalar function defined on P such that

$$v \in C_0^\infty(P; \mathbb{R}^2), f \in C_0^\infty(P; [0, \infty)), \text{ and } f > |v| \text{ on } \operatorname{supp} v. \tag{2.24}$$

For such v, f we define $u[v,f] \colon \mathbb{R}^3 \to \mathbb{R}^3$ to be the rotationally invariant vector field satisfying

$$u[v,f](x_1, x_2, 0) := \left(v_1(x_1, x_2), v_2(x_1, x_2), \sqrt{f(x_1, x_2)^2 - |v(x_1, x_2)|^2} \right) \tag{2.25}$$

for $(x_1, x_2) \in \overline{P}$. Note that such definition immediately gives

$$|u[v,f]| = f. \tag{2.26}$$

Moreover, the definition can we rewritten in a simple, equivalent form using cylindrical coordinates x_1, ρ, ϕ. Namely

$$u[v,f](x_1, \rho, \phi) = v_1(x_1, \rho) \widehat{x}_1 + v_2(x_1, \rho) \widehat{\rho} + \sqrt{f(x_1, \rho)^2 - |v(x_1, \rho)|^2} \, \widehat{\phi}, \tag{2.27}$$

where the cylindrical coordinates are defined using the representation

$$\begin{cases} x_1 = x_1, \\ x_2 = \rho \cos \phi, \\ x_3 = \rho \sin \phi \end{cases}$$

and the cylindrical unit vectors $\widehat{x}_1$, $\widehat{\rho}$, $\widehat{\phi}$ are

$$\begin{cases} \widehat{x}_1(x_1,\rho,\phi) \coloneqq (1,0,0), \\ \widehat{\rho}(x_1,\rho,\phi) \coloneqq (0,\cos\phi,\sin\phi), \\ \widehat{\phi}(x_1,\rho,\phi) \coloneqq (0,-\sin\phi,\cos\phi). \end{cases} \tag{2.28}$$

In particular, for this coordinate system the chain rule gives

$$\begin{aligned} \partial_\rho &= \cos\phi\,\partial_{x_2} + \sin\phi\,\partial_{x_3}, \\ \partial_\phi &= -\rho\sin\phi\,\partial_{x_2} + \rho\cos\phi\,\partial_{x_3}. \end{aligned} \tag{2.29}$$

Clearly, if $\operatorname{supp} v$, $\operatorname{supp} q \subset U$ for some $U \subset P$ then $\operatorname{supp} u[v,q] \subset R(U)$. Moreover, since both v and f have compact support in P and because $f > |v|$ on $\operatorname{supp} v$ (so that $\sqrt{f^2-|v|^2} \in C_0^\infty(P)$) it is clear that $u[v,f] \in C_0^\infty(\mathbb{R}^3,\mathbb{R}^3)$. The vector field $u[v,f]$ enjoys some further useful properties, which we show below.

Lemma 2.5 (Properties of $u[v,f]$)**.**

(i) *The vector field $u[v,f]$ is divergence free if and only if v satisfies*

$$\operatorname{div}(x_2\, v(x_1,x_2)) = 0 \qquad \textit{for all } (x_1,x_2) \in P.$$

(ii) *If $v \equiv 0$ then*

$$\Delta u[0,f](x_1,\rho,\phi) = Lf(x_1,\rho)\widehat{\phi},$$

where

$$Lf(x_1,x_2) \coloneqq \Delta f(x_1,x_2) + \frac{1}{x_2}\partial_{x_2} f(x_1,x_2) - \frac{1}{x_2^2} f(x_1,x_2). \tag{2.30}$$

In particular

$$\Delta u[0,f](x_1,x_2,0) = (0,0,Lf(x_1,x_2)). \tag{2.31}$$

(iii) *For all $x_1,x_2 \in \mathbb{R}$*

$$\partial_{x_3}|u[v,f]|(x_1,x_2,0) = 0. \tag{2.32}$$

Proof. The lemma is a consequence of elementary calculations using cylindrical coordinates. As for (i) recall that the divergence of a vector field u described in cylindrical coordinates as $u = u_1\widehat{x}_1 + u_\rho\widehat{\rho} + u_\phi\widehat{\phi}$ is

$$\operatorname{div} u = \partial_{x_1} u_1 + \frac{1}{\rho}\partial_\rho\left(\rho u_\rho\right) + \frac{1}{\rho}\partial_\phi u_\phi.$$

Thus since $u[v,f]_\phi = \sqrt{f^2-|v|^2}$ does not depend on ϕ we obtain (i).

As for (ii) recall that the Laplacian of any function $F = F(x_1, \rho, \phi)$ is

$$\Delta F = \partial_{x_1 x_1} F + \frac{1}{\rho}\partial_\rho \left(\rho\, \partial_\rho F\right) + \frac{1}{\rho^2}\partial_{\phi\phi} F.$$

Thus, since $u[0,f] = f\widehat{\phi}$ and because the unit vector $\widehat{\phi}$ depends only on ϕ and satisfies $\partial_{\phi\phi}\widehat{\phi} = -\widehat{\phi}$ (recall (2.28)) we obtain

$$\begin{aligned}
\Delta u[0,f] &= \partial_{x_1 x_1} f(x_1,\rho)\widehat{\phi} + \frac{1}{\rho}\partial_\rho \left(\rho\, \partial_\rho f(x_1,\phi)\right)\widehat{\phi} + \frac{f(x_1,\rho)}{\rho^2}\partial_{\phi\phi}\widehat{\phi} \\
&= \partial_{x_1 x_1} f(x_1,\rho)\widehat{\phi} + \frac{1}{\rho}\partial_\rho f(x_1,\phi)\widehat{\phi} + \partial_{\rho\rho} f(x_1,\phi)\widehat{\phi} - \frac{f(x_1,\rho)}{\rho^2}\widehat{\phi} \\
&= Lf(x_1,\rho)\widehat{\phi}.
\end{aligned}$$

In particular, taking $\phi = 0$ gives (2.31).

As for (iii) it is enough to note that, since $|u[v,f]| = f(x_1,\rho)$ is rotationally invariant, the derivative in question is in fact a derivative along a level set of $|u[v,f]|$ (that is along a a circle around the x_1 axis). In other words the relations (2.29) give

$$\partial_{x_3} = \sin\phi\, \partial_\rho + \frac{\cos\phi}{\rho}\partial_\phi \tag{2.33}$$

and so, because $|u[v,f]| = f$ does not depend on ϕ,

$$\partial_{x_3}|u[v,f]| = \sin\phi \left(\partial_\rho f\right),$$

which vanishes when $\phi = 0, \pi$. □

We define $p^*[v,f]\colon \mathbb{R}^3 \to \mathbb{R}$ to be the pressure function corresponding to $u[v,f]$, that is

$$p^*[v,f](x) \coloneqq \int_{\mathbb{R}^3} \sum_{i,j=1}^{3} \frac{\partial_i u_j[v,f](y)\partial_j u_i[v,f](y)}{4\pi|x-y|}\mathrm{d}y, \tag{2.34}$$

and we denote its restriction to $\mathbb{R}^2$ by $p[v,f]$,

$$p[v,f](x_1,x_2) \coloneqq p^*[v,f](x_1,x_2,0). \tag{2.35}$$

It is clear that, since $u[v,f] \in C_0^\infty(\mathbb{R}^3)$,

$$p[v,f] \in C^\infty(\mathbb{R}^2) \tag{2.36}$$

Furthermore, since $u[v,f]$ is rotationally invariant, the same is true of $p^*[v,f]$. In particular, in the same way as in the proof of Lemma 2.5 (iii) above, we obtain that

$$\partial_{x_3} p^*[v,f](x_1,x_2,0) = 0 \quad \text{for all } x_1,x_2 \in \mathbb{R}. \tag{2.37}$$

Similarly,

$$\partial_{x_2} p^*[v,f](x_1,0,x_3) = 0 \quad \text{for all } x_1, x_3 \in \mathbb{R}, \tag{2.38}$$

using the relation

$$\partial_{x_2} = \cos\phi \partial_\rho - \frac{\sin\phi}{\rho}\partial_\phi, \tag{2.39}$$

which is a consequence of (2.29). Thus taking $x_3 = 0$ we obtain

$$\partial_{x_2} p[v,f](x_1,0) = 0 \quad \text{for } x_1 \in \mathbb{R}. \tag{2.40}$$

The function $p[v,f]$ enjoys some further properties, which we state in a lemma.

Lemma 2.6 (Properties of $p[v,f]$)**.** *Let $v = (v_1, v_2), f$ be as in* (2.24)*. Then*

(i) $p[v,f] = p[-v,f]$,

(ii) *if additionally $v_2(\cdot, x_2)$ is odd and $v_1(\cdot, x_2)$, $f(\cdot, x_2)$ are even for each fixed x_2 then $p[v,f]$ is even, that is*

$$p[v,f](x) = p[v,f](-x) \qquad \text{for all } x \in \mathbb{R}^2,$$

(iii) *if $\widetilde{v}, \widetilde{f}$ is another pair satisfying* (2.24) *and such that $f, \widetilde{f}$ have disjoint supports then*

$$p\left[v+\widetilde{v}, f+\widetilde{f}\right] = p[v,f] + p\left[\widetilde{v}, \widetilde{f}\right].$$

Proof. Property (iii) follows directly from the definition (2.34). As for (i) we will show that $p^*[v,f] = p^*[-v,f]$. Substituting (2.28) into (2.27) we obtain

$$\begin{aligned}
u_1[v,f](x_1,\rho,\phi) &= v_1(x_1,\rho),\\
u_2[v,f](x_1,\rho,\phi) &= v_2(x_1,\rho)\cos\phi - \sqrt{f^2-|v|^2}(x_1,\rho)\sin\phi,\\
u_3[v,f](x_1,\rho,\phi) &= v_2\sin\phi + \sqrt{f^2-|v|^2}(x_1,\rho)\cos\phi.
\end{aligned}$$

Thus since for $\phi = 0$ we have $\partial_2 = \partial_\rho$, $\partial_3 = \rho^{-1}\partial_\phi$ (see (2.33), (2.39)) and we obtain

$$\begin{array}{lll}
\partial_1 u_1[v,f] = \partial_{x_1} v_1, & \partial_2 u_1[v,f] = \partial_\rho v_1, & \partial_3 u_1[v,f] = 0,\\
\partial_1 u_2[v,f] = \partial_{x_1} v_2, & \partial_2 u_2[v,f] = \partial_\rho v_2, & \partial_3 u_2[v,f] = -\sqrt{f^2-|v|^2}/\rho,\\
\partial_1 u_3[v,f] = \partial_{x_1}\sqrt{f^2-|v|^2}, & \partial_2 u_3[v,f] = \partial_\rho\sqrt{f^2-|v|^2}, & \partial_3 u_3[v,f] = v_2/\rho,
\end{array} \tag{2.41}$$

from which we immediately see that

$$\partial_i u_j[v,f]\,\partial_j u_i[v,f] = \partial_i u_j[-v,f]\,\partial_j u_i[-v,f]$$

for any choice of $i,j\in\{1,2,3\}$. Summation in i,j gives

$$\sum_{i,j=1}^{3}\partial_i u_j[v,f]\,\partial_j u_i[v,f]=\sum_{i,j=1}^{3}\partial_i u_j[-v,f]\,\partial_j u_i[-v,f]\qquad\text{for }\phi=0,$$

and the rotational invariance of each sum (see (2.21)) gives the equality everywhere in $\mathbb{R}^3$. Consequently, we obtain

$$p^*[-v,f]=p^*[v,f],$$

as required.

As for (ii), we will show that

$$\left(\sum_{i,j=1}^{3}\partial_i u_j[v,f]\,\partial_j u_i[v,f]\right)(x_1,\rho)=\left(\sum_{i,j=1}^{3}\partial_i u_j[v,f]\,\partial_j u_i[v,f]\right)(-x_1,\rho)\qquad(2.42)$$

for $x_1\in\mathbb{R}$, $\rho>0$, where we skipped the ϕ in the variable (recall that this sum is independent of ϕ due to the rotational invariance (2.21)). In other words, the sum

$$\sum_{i,j=1}^{3}\partial_i u_j[v,f]\,\partial_j u_i[v,f]$$

is an even function (recall that in cylindrical coordinates $\rho=\sqrt{x_2^2+x_3^2}$ takes the same value for x and $-x$) and so consequently $p^*[v,f]$ is even on $\mathbb{R}^3$ (by definition, see (2.34)). Then in particular $p[v,f]$ is even on $\mathbb{R}^2$, as required. Thus it suffices to show (2.42).

To this end take $(-x_1,\rho,0)$ as the variable in (2.41) to obtain the same expressions as in the case of $(x_1,\rho,0)$, except for the diagonal expressions, which are now of the opposite sign. This, however, makes no change to the sum

$$\sum_{i,j=1}^{3}\partial_i u_j[v,f]\,\partial_j u_i[v,f],$$

that is

$$\left(\sum_{i,j=1}^{3}\partial_i u_j[v,f]\,\partial_j u_i[v,f]\right)(x_1,\rho,0)=\left(\sum_{i,j=1}^{3}\partial_i u_j[v,f]\,\partial_j u_i[v,f]\right)(-x_1,\rho,0),$$

and thus (2.42) follows from the rotational invariance. $\square$

2.2.4 A structure on $U \Subset P$

The definitions in the previous section give rise to a way of defining a smooth, divergence-free velocity field u supported on $R(\overline{U})$, for $U \Subset P$. The following notion of a structure is a part of our simplified approach to the constructions.

Definition 2.7. *A* structure on $U \Subset P$ *is a triple* (v, f, ϕ), *where* $v \in C_0^\infty(U;\mathbb{R}^2)$, $f \in C_0^\infty(P;[0,\infty))$, $\phi \in C_0^\infty(U;[0,1])$ *are such that* $\operatorname{supp} f = \overline{U}$,

$$\operatorname{supp} v \subset \{\phi = 1\}, \qquad \operatorname{div}\left(x_2\, v(x_1,x_2)\right) = 0 \text{ in } U \qquad \text{and}$$
$$f > |v| \quad \text{in } U \qquad \text{with} \qquad Lf > 0 \text{ in } U \setminus \{\phi = 1\}.$$

Note that (av, f, ϕ) is a structure for any $a \in (-1,1)$ whenever (v, f, ϕ) is.

Furthermore, given (v, f, ϕ), a structure on U, the velocity field $u[v,f]$ is divergence free and is supported in $R(\overline{U})$. Moreover in $R(\{\phi < 1\})$

$$u[v,f] \cdot \Delta u[v,f] \geq 0 \tag{2.43}$$

and

$$u[v,f] \cdot \nabla q = 0 \tag{2.44}$$

for any rotationally symmetric function $q\colon \mathbb{R}^3 \to \mathbb{R}$. This last property is particularly useful when taking $q := |u[v,f]|^2 + 2p[v,f]$ as in this way the left-hand side of (2.44) is of the same form as one of the terms in the Navier–Stokes inequality (2.6). In order to see (2.43), (2.44) first note that, due to the rotational invariance it is enough to verify that

$$u[v,f](x_1,x_2,0) \cdot \Delta u[v,f](x_1,x_2,0) \geq 0$$

and

$$u[v,f](x_1,x_2,0) \cdot \nabla q(x_1,x_2,0) = 0$$

for $(x_1,x_2) \in \{\phi < 1\}$ (recall (2.21)). Since $v = 0$ in $\{\phi < 1\}$ we have $u[v,f](x_1,x_2,0) = (0,0,f(x_1,x_2))$ (recall (2.25)), and so obtain the first of the above properties by writing

$$u[v,f](x_1,x_2,0) \cdot \Delta u[v,f](x_1,x_2,0) = f(x_1,x_2) Lf(x_1,x_2) \geq 0 \tag{2.45}$$

where we used Lemma 2.5 (ii). The second property follows in the same way by noting that $\partial_{x_3} q(x_1,x_2,0) = 0$ (as a property of a rotationally invariant function, which can be obtained in the same way as (2.32)).

Furthermore, note that given U, the L^∞ norm of derivatives of $u[v,f]$ can be bounded above by a constant depending only on $W^{1,\infty}$ norm of v and f, that is

$$\|\nabla u[v,f]\|_{L^\infty} \le C\left(\|v\|_{W^{1,\infty}}, \|f\|_{W^{1,\infty}}\right). \tag{2.46}$$

Note also that the constant depends on U only in terms of its distance from the x_1 axis.

2.2.5 A recipe for a structure

In the rest of this chapter we will only consider functions v, f and sets $U \Subset P$ such that for some ϕ the triple (v,f,ϕ) is a structure on U. Moreover, we will only consider sets U in the shape of a rectangle or a "rectangular ring", that is $V \setminus \overline{W}$, where V, W are open rectangles and $W \Subset V$. One can construct structures on such sets in a generic way, which we now describe.

First construct $v \in C_0^\infty(U,\mathbb{R}^2)$ satisfying $\mathrm{div}\,(x_2\, v(x_1,x_2)) = 0$ for all $(x_1,x_2) \in U$. For this it is enough to take a mollification of w and divide it by x_2, where $w\colon U \to \mathbb{R}^2$ is a compactly supported and weakly divergence free vector field, that is $\int_P w\cdot\nabla\psi = 0$ for every $\psi \in C_0^\infty(P;\mathbb{R})$. Indeed, then the mollification of w is divergence-free and thus $\mathrm{div}\,(x_2\, v(x_1,x_2)) = 0$. As for the construction of w take, for example,

$$w := (x_2 - 3, x_1)\chi_{1<|x-(0,3)|<2},$$

where χ denotes the indicator function, see Fig. 2.3. Note that w is weakly divergence free due to the fact that $w\cdot n$ vanishes on the boundary of the support of w, where n denotes the respective normal vector to the boundary. Alternatively, define w to be a "regionwise" constant velocity field

$$w := \begin{cases} (1,0) & \text{in } R_1,\\ (0,1) & \text{in } R_2,\\ (-1,0) & \text{in } R_3,\\ (0,-1) & \text{in } R_4, \end{cases}$$

where R_1, R_2, R_3, R_4 are arranged as in Fig. 2.3.

An integration by parts and the use of the crucial property of $w\cdot n$ being continuous across the boundary between each pair of neighbouring regions R_1, R_2, R_3, R_4, $P \setminus \bigcup_i R_i$ immediately shows that such a w is weakly divergence free. An advantage of

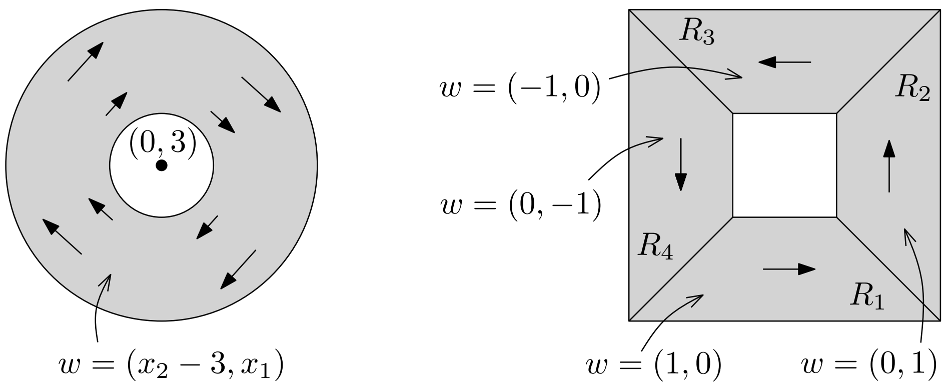

Figure 2.3: Constructing compactly supported, weakly divergence free vector field w.

such a definition of w (as compared to the previous one) is that it can be "stretched geometrically" in a sense that given $\varepsilon > 0$ one can modify w to obtain $w = (1, 0)$ in any given strict subset of P and $|w| < \varepsilon$ whenever v has a direction other than $(1, 0)$, see Fig. 2.4. We will later see an important sharpening of this observation (see Lemma 2.14).

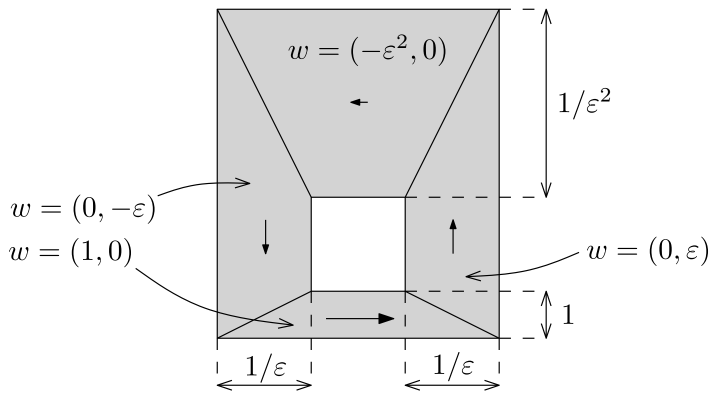

Figure 2.4: Deforming the vector field w.

Secondly, let $\mu, \eta > 0$ be such that $\mu > \|v\|_\infty$ and $\operatorname{supp} v \subset U_\eta$, where

$$U_\eta \coloneqq \{x \in U : \operatorname{dist}(x, \partial U) > \eta\}$$

denotes the *η-subset* of U, and let $f \in C_0^\infty(P; [0, \infty))$ be a certain cut-off function (in U) that has a particular behaviour near ∂U. Namely, let f be given by the following theorem.

Theorem 2.8. *Let $U \Subset P$ be an open set that is in the shape of a rectangle or $U = V \setminus \overline{W}$ for some open rectangles $V, W \Subset P$ with $W \Subset V$. Given $\eta > 0$ there exists*

$\delta \in (0,\eta)$ and $f \in C_0^\infty(P,[0,1])$ such that

$$\operatorname{supp} f = \overline{U}, \quad f > 0 \text{ in } U \text{ with } f = 1 \text{ on } U_\eta$$

and

$$Lf > 0 \quad \text{in } U \setminus U_\delta.$$

The proof of the theorem is elementary in nature, but requires some technicalities, in particular a generalised form of the Mean Value Theorem (see Lemma 2.16). We prove the theorem in Appendix 2.5.1 (see Lemma 2.18 for the case of a rectangle and Lemma 2.19 for the case of a rectangular ring).

Finally, having defined v and f, one can simply take any cut-off function $\phi \in C_0^\infty(U;[0,1])$ such that $\phi = 1$ on U_δ. Thus we obtain a structure (v,f,ϕ) on U. Note that the choice of (sufficiently large) $\mu = \|f\|_\infty$ is arbitrary.

2.2.6 The pressure interaction function $F[v,f]$

As in the case of the notion of a structure (v,f,ϕ) on a set $U \Subset P$, we simplify Scheffer's approach by introducing the notion of a *pressure interaction function corresponding to* U,

$$F[v,f] \coloneqq \nabla p[0,f] - \nabla p[v,f], \tag{2.47}$$

where ∇ denotes the two-dimensional gradient. Note that $F[v,f]$ depends on the structure (v,f,ϕ) on U, and thus a set $U \Subset P$ can possibly have more than one pressure interaction function. It is not clear whether $F[v,f]$ has any physical interpretation, but this is the tool that will form certain interactions between subsets of P (see the comments following Theorem 2.12), and we will see later that, in a sense, the strength of this interaction can be adjusted by manipulating the subsets and their corresponding structures (see the comments following (2.97) and the subsequent Sections 2.4.2-2.4.5).

We now show that $F[v,f]$ enjoys a number of useful properties, which include estimates of its size at points near the x_1 axis.

Lemma 2.9 (Properties of the pressure interaction function $F[v,f]$)**.** *Let (v,f,ϕ) be a structure on some $U \Subset P$ such that $v_1 \not\equiv 0$. Then the pressure interaction function $F \coloneqq F[v,f]$ satisfies*

(i) *$F \in C^\infty(\mathbb{R}^2;\mathbb{R}^2)$ and*

$$\lim_{x_1 \to \pm\infty} x_1^4 F_1(x_1,0) = \frac{\pm 9}{4\pi} \int_{\mathbb{R}^3} v_1^2\left(y_1, \sqrt{y_2^2+y_3^2}\right) \mathrm{d}y =: \pm D.$$

(ii) F_1 *restricted to the* x_1 *axis attains a positive maximum, that is there exists* $B > 0$, $A \in \mathbb{R}$ *such that*

$$B = F_1(A, 0) = \max_{x_1 \in \mathbb{R}} F_1(x_1, 0).$$

(iii) *There exists* $C > 0$ *such that*

$$|F(x)| \leq C/|x|^4, \qquad |\nabla F(x)| \leq C/|x|^5 \quad \text{for } x \in \mathbb{R}^2.$$

(iv) $F_2(x_1, 0) = 0$ *for* $x_1 \in \mathbb{R}$.

(v) *Let*

$$\kappa := 10^4 C/D. \tag{2.48}$$

There exists $N > 0$ *such that for* $n \geq N$

$$|x_1 - n| < \kappa,\ |x_2| < 1 \text{ implies } |F_1(x_1, x_2) - n^{-4}D| \leq 0.001 n^{-4} D.$$

Proof. Claim (ii) follows from (i) and the assumption $v_1 \not\equiv 0$. As for (i), the smoothness of F follows directly from the fact that (v, f, ϕ) is a structure on U, and the limiting property as $x_1 \to \pm\infty$ follows by using (2.23), from which we obtain

$$\begin{aligned} \lim_{x_1 \to \infty} x_1^4 F_1(x_1, 0) &= \frac{3}{4\pi} \int_{\mathbb{R}^3} \left(|u[0, f]|^2 - 3(u_1[0, f])^2 - |u[v, f]|^2 + 3u_1^2[v, f])^2 \right) \mathrm{d}y \\ &= \frac{9}{4\pi} \int_{\mathbb{R}^3} v_1(R^{-1}(y)) \mathrm{d}y, \end{aligned}$$

where we also used the facts $|u[v, f](y)| = f(R^{-1}(y))$, $u_1[v, f](y) = v_1(R^{-1}(y))$ (see (2.27)). The case of the limit $x_1 \to -\infty$ is similar.

Claim (iii) follows from the decay properties of the pressure function, see (2.22). Claim (iv) follows directly from (2.38).

As for (v), suppose that $|x_1 - n| < \kappa$. Then for sufficiently large n (and so also x_1)

$$|n^4 - x_1^4| = |n^2 + x_1^2|\, |n + x_1|\, |n - x_1| \leq \widetilde{C} |x_1|^3$$

for some $\widetilde{C} > 0$ (depending on κ). Thus

$$|n^4 F_1(x_1, 0) - D| \leq |n^4 - x_1^4|\, |F_1(x_1, 0)| + |x_1^4 F_1(x_1, 0) - D| \leq \widetilde{C} C |x_1|^{-1} + |x_1^4 F_1(x_1, 0) - D|.$$

Since taking n large makes x_1 large as well, we see from (i) that for sufficiently large n $|n^4 F_1(x_1, 0) - D| \leq 0.0005 D$, that is

$$|F_1(x_1, 0) - n^{-4} D| \leq 0.0005 n^{-4} D. \tag{2.49}$$

Moreover, the Mean Value Theorem gives for $|x_2| < 1$ and sufficiently large n

$$|F_1(x_1,x_2) - F_1(x_1,0)| \le |x_2|\,|\nabla F_1(x_1,\xi)| \le C|x_1|^{-5} \le 0.0005 n^{-4} D,$$

where $\xi \in (0,1)$. The claim follows from this and (2.49). $\square$

2.3 The setting

In this section we define constants $T > 0$, $\nu_0 > 0$, $\tau \in (0,1)$, $z \in \mathbb{R}^3$, the set G and the vector field u which were required in the sketch proof in Section 2.1. The definition is based on a certain geometric setting which we formalise here in the notion of the *geometric arrangement*.

By the *geometric arrangement* we mean a pair of open sets $U_1, U_2 \Subset P$ together with the corresponding structures (v_1, f_1, ϕ_1), (v_2, f_2, ϕ_2) (recall Definition 2.7) such that $\overline{U_1} \cap \overline{U_2} = \emptyset$ and, for some $T > 0$, $\tau \in (0,1)$, $z \in \mathbb{R}^3$,

$$f_2^2 + T v_2 \cdot F[v_1, f_1] > |v_2|^2 \quad \text{in } U_2, \tag{2.50}$$

$$f_2^2(y) + T v_2(y) \cdot F[v_1, f_1](y) > \tau^{-2} \left(f_1(R^{-1}x) + f_2(R^{-1}x) \right)^2 \tag{2.51}$$

for all $x \in G := R\left(\overline{U_1} \cup \overline{U_2}\right)$, where

$$y = R^{-1}(\Gamma(x)) \tag{2.52}$$

(recall $\Gamma(x) = \tau x + z$), and

$$\Gamma(G) \subset G. \tag{2.53}$$

Before defining the remaining constant ν_0 and vector field u, we comment on the notion of the geometric arrangement in an informal way.

Recall from Section 2.1 that we aim to find a vector field u, which is defined on the time interval $[0,T]$, that satisfies the NSI (2.6) as well as admits the gain in magnitude (2.7). We want to obtain the gain via the term $u \cdot \nabla p$, which we now discuss. We will construct u in a way that, at time $t = 0$

$$u(0) \approx u[v_1, f_1] + u[v_2, f_2],$$

and at time $t = T$

$$u(T) \approx u[v_1, f_1] + u\left[v_2, \sqrt{f_2^2 + Tv_2 \cdot F[v_1, f_1]}\right]. \tag{2.54}$$

In other words, u is to consist of two disjointly supported (in space) vector fields. The first of them will be supported in $R(U_1)$ and its absolute value (that is f_1) will remain (approximately) constant through the time interval $[0, T]$. The second of them will be supported in U_2 and its absolute value will change in time from f_2 to (approximately) $\sqrt{f_2^2 + Tv_2 \cdot F[v_1, f_1]}$.

At this point it is clear that the requirement (2.50) is necesseary for the right-hand side of (2.54) to be well-defined (recall (2.24)). Furthermore, in light of the property $|u[v, f]| = f$ (valid for any (admissible) v, f, recall (2.26)) we see that the requirement (2.51) means simply that

$$|u(\Gamma(x), T)|^2 \gtrsim \tau^{-2}|u(x, 0)|^2.$$

By writing "approximately" (or $\approx$, $\gtrsim$) we mean "very close in the $L^\infty(\mathbb{R}^3)$ norm". We note such an approximate sense will be made rigorous below by using continuity arguments as well as the facts that the inequalities in (2.50) and (2.51) are sharp ("$>$") and the supports of the functions appearing on their right-hand sides are compact.

It remains to ask why the term "$Tv_2 \cdot F[v_1, f_1]$" is chosen to achieve the gain in magnitude.

A rough answer to this question is: because (1) the pressure interaction function has a certain property that allows us to magnify it and because (2) that this one of the very few degrees of freedom allowed by the Navier–Stokes inequality. We have already observed (1) in Lemma 2.9 (particularly part (ii)), and we will see the full power of it in the construction of the geometric arrangement in Section 2.4. As for (2), recall the NSI (2.6),

$$\partial_t |u|^2 \le -u \cdot \nabla \left(|u|^2 - 2p \right) + 2\nu\, u \cdot \Delta u.$$

We illustrate the reason for the term "$Tv_2 \cdot F[v_1, f_1]$" by the following thought experiment. Suppose that

$$u = u[v_1, f_1] + u[v_2, f_2] \tag{2.55}$$

and take a close look at the terms appearing on the right-hand side of the NSI above, where we ignore, for a moment, the time dependence. First of all, the pressure function

p, is given by $p^*[v_1, f_1] + p^*[v_2, f_2]$ (recall Lemma 2.6 (iii)). Thus, since both u and p are rotationally invariant, so are all the terms on the right-hand side of the NSI (recall (2.21)). Thus it is sufficient to look only at points x of the form $(x_1, x_2, 0)$, $(x_1, x_2) \in \mathbb{R}^2$. At such points the right-hand side of the NSI takes the form

$$-\{(v_1 + v_2) \cdot \nabla\} \left(f_1^2 + f_2^2 + 2p[v_1, f_1] + 2p[v_2, f_2]\right) + 2\nu u(\cdot, 0) \cdot \Delta u(\cdot, 0), \qquad (2.56)$$

where $v_1 = (v_{11}, v_{12})$, $v_2 = (v_{21}, v_{22})$ and $\nabla = (\partial_1, \partial_2)$ now denotes the two-dimensional gradient; recall also that $p[v_i, f_i] = p^*[v_i, f_i](\cdot, 0)$ (see (2.35)). Observe that the ∂_3 derivative does not appear since both u and p are rotationally invariant (and so ∂_3 is a derivative along a level set, recall (2.32) and (2.37)).

The last term in (2.56) will not play any significant role in our analysis; we will treat it as an error term. In fact, we already know how to deal with this term at points (x_1, x_2) such that $(\phi_1 + \phi_2)(x_1, x_2) < 1$ (recall that ϕ_1, ϕ_2 play the role of a cutoff function in the structures (v_1, f_1, ϕ_1), (v_2, f_2, ϕ_2), respectively; see Definition 2.7). Indeed, at such points $v_1 = v_2 = 0$, and so (2.56) becomes

$$2\nu(f_1\, Lf_1 + f_2\, Lf_2) \geq 0$$

(recall (2.45)). This non-negativity will turn out sufficient for the NSI (see (2.78) below for details), while at points (x_1, x_2) such that $(\phi_1+\phi_2)(x_1, x_2) = 1$ we will use continuity arguments to take ν sufficiently small (see (2.68) and (2.81) below for details).

As for the first term in (2.56), we will be interested in interactions between $u[v_1, f_1]$ and $u[v_2, f_2]$ (this is the reason why the geometric arrangement consists of two sets U_1, U_2 and their corresponding structures) and so from the terms in (2.56) we are concerned with the mixed terms of the form

(something supported in U_i) (a function "based on" U_j and its structure)

for $i, j = 1, 2$, $i \neq j$, namely with the terms

$$-v_i \cdot \nabla f_j^2 \qquad \text{and} \qquad -2v_i \cdot \nabla p[v_j, f_j],$$

$i, j = 1, 2$, $i \neq j$. Note that the first of such terms vanishes since v_i and f_j have disjoint supports. As for the second one, we will be manipulating only the terms with the "∂_1" derivative since we are only able to control this derivative of the pressure function (which comes, fundamentally, from the property (2.23) and from our choice of picking

Ox_1 as the axis of symmetry; this fact has been explored further in Lemma 2.9). In fact, we aim to construct the geometric arrangement in such a way that

$$-v_{21}\partial_1(p[v_1,f_1]-p[0,f_1]) = v_{21}F_1[v_1,f_1] \qquad \text{is large}$$

in a certain region of U_2 that is close to the Ox_1 axis (see Section 2.4.1 for a wider discussion of this issue). In other words we will try to, in a sense, magnify the influence of U_1 (and its structure) onto U_2 (and its structure).

We now discuss the issue of time dependence, which will lead us to the term "$Tv_2 \cdot F[v_1,f_1]$" (which plays a crucial role in the geometric arrangement). In fact, instead of the naive candidate (2.55), we will actually consider a time dependent vector field of the form

$$u(t) = u[v_{1,t}, f_{1,t}] + u[v_{2,t}, f_{2,t}],$$

where $v_{i,t}$, $f_{i,t}$ are certain time dependent extensions of v_i, f_i, respectively (see (2.72) and (2.71) below for the exact formula), which are chosen so that

$$\begin{aligned} |u(\cdot,0,t)|^2 = f_{1,t}^2 + f_{2,t}^2 &= f_1^2 + f_2^2 + (\text{something small, negative and linear in } t) \\ &- \int_0^t (v_{1,s}+v_{2,s}) \cdot \nabla (f_{1,s}^2 + f_{2,s}^2 + 2p[v_{1,s},f_{1,s}] + 2p[v_{2,s},f_{2,s}]) \mathrm{d}s, \end{aligned} \tag{2.57}$$

(We write "$(\cdot,0,t)$" to articulate that we restrict ourselves to points (x,t) of the form $(x_1,x_2,0,t)$.) Note that by taking ∂_t we obtain

$$\begin{aligned} \partial_t |u(\cdot,0,t)|^2 &= -\,(\text{something small}) \\ &\quad - (v_{1,t}+v_{2,t}) \cdot \nabla (f_{1,t}^2 + f_{2,t}^2 + 2p[v_{1,t},f_{1,t}] + 2p[v_{2,t},f_{2,t}]), \\ &= -\,(\text{something small}) - u(\cdot,0,t)\cdot \nabla(|u(\cdot,0,t)|^2 + 2p(\cdot,0,t)). \end{aligned}$$

Here, the small term will be used in the continuity argument to absorb the Laplacian term, $\nu u \cdot \Delta u$ (compare with (2.56)), see (2.78) and (2.81) for details. In other words, the time dependent extensions $v_{i,t}$, $f_{i,t}$ $(i=1,2)$ will be chosen such that, by construction, we will obtain the NSI.

In particular, we will choose

$$v_{i,t} = a_i(t) v_i, \qquad i = 1,2,$$

where $a_1, a_2 \in C^\infty(\mathbb{R}; [-1,1])$ are certain *oscillatory processes*, which are discussed in detail in Section 2.3.3 below. The oscillatory process will have two remarkable features.

The first is that

$$\int_0^t a_i(s) v_i \cdot \nabla f_{i,s} \mathrm{d}s \approx 0, \qquad \text{uniformly in } i=1,2, t\in[0,T],$$

and it will be a simple consequence of high oscillations of a_1, a_2. The second remarkable feature is that they enable us to pick from all the terms

$$\int_0^t a_i(s) v_i \cdot \nabla p[a_j(s)v_j, f_{j,s}] \mathrm{d}s, \qquad i,j\in\{1,2\}$$

any of the terms

$$\int_0^t v_i \cdot \nabla p[v_j, f_{j,s}],$$

provided we subtract $p[0,f_{j,s}]$. To be more precise for any choice of indices $i_0, j_0 \in \{1,2\}$ there exist oscillatory processes $a_1, a_2 \in C^\infty(\mathbb{R};[-1,1])$ such that

$$\sum_{i,j=1}^2 \int_0^t a_i(s) v_i \cdot \nabla p[a_j(s)v_j, f_{j,s}] \mathrm{d}s \approx \int_0^t v_{i_0} \cdot \nabla (p[v_{j_0}, f_{j_0,s}] - p[0, f_{j_0,s}]) \mathrm{d}s \quad \text{for } t\in[0,T].$$

Therefore, choosing $(i_0, j_0) \coloneqq (2,1)$ (since we are interested in the influence of U_1 onto U_2) we obtain that the integral on the right-hand side of (2.57) is approximately

$$\int_0^t v_2 \cdot \nabla (p[v_1, f_{1,s}] - p[0, f_{1,s}]) \mathrm{d}s = -\int_0^t v_2 \cdot F[v_1, f_{1,s}] \mathrm{d}s \quad \text{for all } t\in[0,T]$$

On the other hand, we will make a choice of $f_{1,s}$ that is, roughly speaking, very slowly depending on s, so that the last integral is approximately

$$t\, v_2 \cdot F[v_1, f_1].$$

That is, we will choose the oscillatory processes a_1, a_2 and the time-dependent extensions of f_1, f_2 such that, except for the expression of $|u(t)|$ given (approximately) by (2.57), we will obtain, at the same time, another one:

$$|u(\cdot,0,t)|^2 \approx f_1^2 + f_2^2 + t\, v_2 \cdot F[v_1, f_1] \quad t\in[0,T]. \tag{2.58}$$

This explains (by taking $t=T$) the appearance of the term $Tv_2 \cdot F[v_1, f_1]$ in the geometric arrangement.

To sum up the above heuristic discussion, based on any disjoint sets U_1, U_2 and their corresponding structures (v_1, f_1, ϕ_1), (v_2, f_2, ϕ_2) we can find a way of prescribing the time dependence (on any time interval) such that the NSI is satisfied (by prescribing behaviour in time, in particular by the oscillatory processes) and that $|u(t)|$ is approximately as in (2.58), which in turn we are able to magnify (at least in some region of the support) by arranging U_1, U_2 (and the corresponding structures) and defining $T > 0$ appropriately; namely by constructing the geometric arrangement.

The construction of the geometric arrangement (which is sketched in Fig. 2.5) is a nontrivial matter and it is in fact the heart of the proof of Theorem 2.1. We present it in Section 2.4.

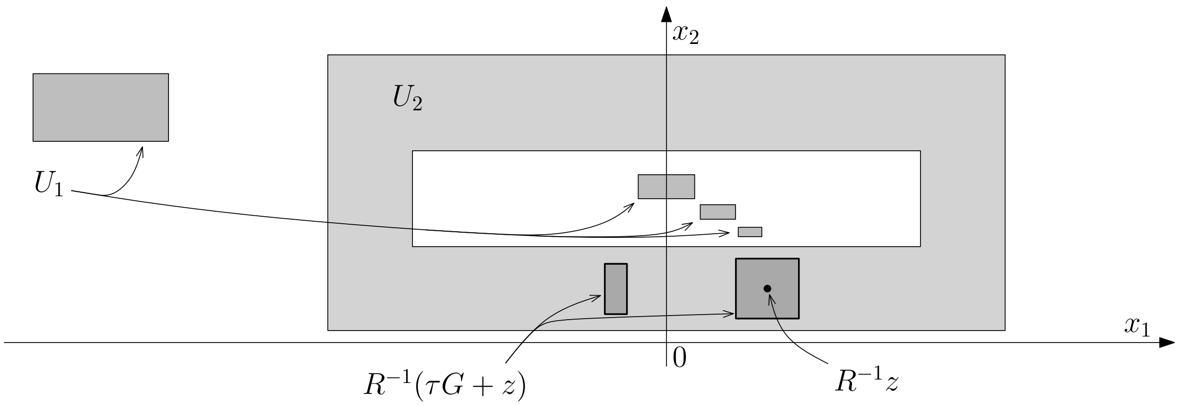

Figure 2.5: A sketch of the geometric arrangement, see Section 2.4 for details. Note that the inclusion $R^{-1}(\tau G + z) \subset U_2 \subset R^{-1}(G)$, which is illustrated on this sketch, implies (2.53). Proportions are not conserved on this sketch.

In the remainder of this section, we assume that the geometric arrangement is given and we apply the strategy above, but in a rigorous way. Namely we obtain ν_0 and u (the remaining constants $T > 0$, $\tau \in (0,1)$, $z \in \mathbb{R}^3$ and the compact set $G \subset \mathbb{R}^3$, which were required in the sketch argument in Section 3.2, are given by the geometric arrangement).

We note that, except for the need of rigorous presentation (in the remainder of this section as well as in Section 2.4, where we construct the geometric arrangement), it is also rather pleasing to observe all components of the construction fit together.

Furthermore, we will not be using the notation $f_{i,t}$ (to denote the time extension of f_i, $i = 1, 2$), but rather $h_{i,t}$ (the time extension of f_i) and $q^k_{i,t}$ (an approximation of $h_{i,t}$, where k is large).

Let $\theta > 0$ be sufficiently small such that

$$f_2^2(y) + Tv_2(y) \cdot F[v_1, f_1](y) > \tau^{-2} \left(f_1(R^{-1}x) + f_2(R^{-1}x) \right)^2 + 2\theta \tag{2.59}$$

for $x \in G$. Such a choice is possible by continuity since the inequality in (2.51) is strict and G is compact.

Let $h\colon P \times [0,T] \to [0,\infty)$ be defined by

$$h_t = h_{1,t} + h_{2,t} \tag{2.60}$$

(recall we use the convention $h_t(\cdot) \equiv h(\cdot, t)$), where

$$h_{1,t}^2 \coloneqq f_1^2 - 2t\delta\phi_1, \tag{2.61}$$

$$h_{2,t}^2 \coloneqq f_2^2 - 2t\delta\phi_2 + \int_0^t v_2 \cdot F[v_1, h_{1,s}] \, \mathrm{d}s. \tag{2.62}$$

Thus $h_{i,t}$ is a time dependent modification of f_i, $i = 1, 2$, such that $h_{i,t} = f_i$ outside $\operatorname{supp} \phi_i$ (recall $\operatorname{supp} v_2 \subset \operatorname{supp} \phi_2$, see Definition 2.7). Here $\delta > 0$ is a fixed, small number given by the following lemma.

Lemma 2.10 (properties of functions $h_{1,t}$, $h_{2,t}$)**.** *There exists $\delta > 0$ (sufficiently small) such that $h_1, h_2 \in C^\infty(P \times (-\delta, T+\delta); [0,\infty))$,*

$$(v_i, h_{i,t}, \phi_i) \text{ is a structure on } U_i \qquad \text{for } t \in (-\delta, T+\delta), i = 1, 2, \tag{2.63}$$

and

$$h_{2,T}^2(y) > \tau^{-2} \left(f_1(R^{-1}x) + f_2(R^{-1}x) \right)^2 + \theta \qquad \text{for } x \in R\left(\overline{U_1} \cup \overline{U_2}\right). \tag{2.64}$$

Proof. For $h_{1,t}$ note that since $f_1 > 0$ in U_1 we can take $\delta \in (0,1)$ such that

$$\delta < \min_{\operatorname{supp} \phi_1} \left| f_1^2 - |v_1|^2 \right| / 2(T+2)$$

to obtain

$$h_{1,t} > |v_1| \qquad \text{in } \operatorname{supp} \phi_1 \text{ for } t \in [-1, T+1].$$

Thus, since $h_{1,t} = f_1$ outside $\operatorname{supp} \phi_1$,

$$h_{1,t} > |v_1| \geq 0 \qquad \text{in } U_1 \text{ for } t \in (-\delta, T+\delta).$$

Hence, since both f_1 and ϕ_1 are smooth on P we immediately obtain the required smoothness of h_1 and that $(v_1, h_{1,t}, \phi_1)$ is a structure on U_1 for all $t \in (-\delta, T+\delta)$.

As for $h_{2,t}$, suppose for the moment that $\delta = 0$. Then $h_{1,t} = f_1$ and so

$$h_{2,t}^2 = f_2^2 + t\, v_2 \cdot F[v_1, f_1]. \tag{2.65}$$

This means that

$$h_{2,0}^2 = f_2^2 \quad \text{and} \quad h_{2,T}^2 = f_2^2 + T v_2 \cdot F[v_1, f_1] \quad \text{if } \delta = 0. \tag{2.66}$$

Using the fact that (v_2, f_2, ϕ_2) is a structure on U_2 and (2.50), we see that both of the above functions are greater than $|v_2|^2$ in U_2. In particular they are greater than $|v_2|^2$ on the compact set $\operatorname{supp} \phi_2$. Since $h_{2,t}$ in (2.65) depends linearly on t we thus obtain

$$h_{2,t} > |v_2| \quad \text{in } \operatorname{supp} \phi_2, \text{ for } t \in [0, T] \quad \text{if } \delta = 0.$$

Therefore, since h_2 depends continuously on δ, we obtain

$$h_{2,t} > |v_2| \quad \text{in } \operatorname{supp} \phi_2, \text{ for } t \in [0, T] \quad \text{if } \delta > 0 \text{ is sufficiently small.}$$

Thus, by continuity in time, this property holds also for t belonging to an open interval containing $[0, T]$. Taking δ smaller we can take this open interval to be $(-\delta, T + \delta)$. Thus, recalling that $h_{2,t} = f_2$ outside $\operatorname{supp} \phi_2$ we obtain

$$h_{2,t} > |v_2| \geq 0 \quad \text{in } U_2, \text{ for } t \in (-\delta, T + \delta) \quad \text{if } \delta > 0 \text{ is sufficiently small.}$$

As in the case of $h_{1,t}$ this immediately gives the required regularity of h_2 and that $(v_2, h_{2,t}, \phi_2)$ is a structure on U_2.

As for (2.64) note that (2.66) gives in particular

$$h_{2,T}^2 = f_2^2 + T v_2 \cdot F[v_1, f_1] \quad \text{in } \operatorname{supp} \phi_2 \qquad \text{if } \delta = 0,$$

and so for sufficiently small $\delta > 0$

$$h_{2,T}^2 \geq f_2^2 + T v_2 \cdot F[v_1, f_1] + \theta \qquad \text{in } \operatorname{supp} \phi_2.$$

Since $h_{2,T} = f_2$ and $v_2 = 0$ outside $\operatorname{supp} \phi_2$, we trivially obtain the above inequality outside $\operatorname{supp} \phi_2$, and so (2.64) follows from this and (2.59). □

Note that (2.63) gives in particular that

$$(av_i, h_{i,t}, \phi_i) \text{ is a structure on } U_i \qquad \text{for } t \in (-\delta, T + \delta), i = 1, 2, \tag{2.67}$$

for any $a \in [-1, 1]$ ($t \in (-\delta, T + \delta)$, $i = 1, 2$).

At this point we fix $\nu_0 > 0$ sufficiently small such that

$$\nu_0 \,|u[av_i, h_{i,t}] \cdot \Delta u[av_i, h_{i,t}]| < \delta/8 \quad \text{in } \mathbb{R}^3 \tag{2.68}$$

for $a \in [-1,1]$, $i = 1,2$, $t \in [0,T]$.

Having constructed the time dependent functions $h_{1,t}, h_{2,t}$ and having fixed ν_0, we now construct u.

Proposition 2.11. *There exist $\eta \in (0,\delta)$ and $u \in C^\infty(\mathbb{R}^3 \times (-\eta, T+\eta); \mathbb{R}^3)$ such that*

(i) $\operatorname{supp} u(t) = G$ *and* $\operatorname{div} u(t) = 0$ *for* $t \in (-\eta, T+\eta)$,

(ii) $|u(x,0)| = h_0(R^{-1}x)$ *and* $|\,|u(x,t)|^2 - h_t(R^{-1}x)^2| < \theta$ *for all* $x \in \mathbb{R}^3$, $t \in [0,T]$,

(iii) *the Navier–Stokes inequality*

$$\partial_t |u|^2 \le -u \cdot \nabla \left(|u|^2 + 2p \right) + 2\nu\, u \cdot \Delta u$$

holds in $\mathbb{R}^3 \times [0,T]$ *for all* $\nu \in [0,\nu_0]$ *where* p *is the pressure function corresponding to* u.

Note that η, u given by the proposition satisfy all the properties required in Section 2.1. Among those only (2.7) is nontrivial; this follows from (ii) and (2.64) by writing

$$\begin{aligned}
|u(\Gamma(x),T)|^2 &\ge h_T(R^{-1}(\Gamma(x))^2 - \theta \\
&\ge h_{2,T}(R^{-1}(\Gamma(x))^2 - \theta \\
&> \tau^{-2} \left(f_1(R^{-1}x) + f_2(R^{-1}x) \right)^2 \\
&= \tau^2 h_0(R^{-1}(x))^2 \\
&= \tau^{-2} \,|u(x,0)|^2
\end{aligned}$$

for $x \in G$ (the case $x \notin G$ is trivial), where we also used (2.53) in the second step. The rest of the properties follow directly from (i), (iii). It remains to prove Proposition 2.11. The proof is separated into three steps, which we present in Sections 2.3.1-2.3.3 below.

2.3.1 The construction of u

We will find u (a solution to Proposition 2.11) that is rotationally invariant (see (2.20)). For such a vector field (ii) is equivalent to

$$|u(x,0,0)| = h_0(x), \quad \text{and} \quad \left|\,|u(x,0,t)|^2 - h_t(x)^2\right| < \theta \quad \text{for } x \in P, t \in [0,T] \tag{2.69}$$

and (iii) is equivalent to

$$\partial_t |u(x,0,t)|^2 \le -u(x,0,t)\cdot\nabla\left(|u(x,0,t)|^2+2p(x,0,t)\right)+2\nu\, u(x,0,t)\cdot\Delta u(x,0,t) \tag{2.70}$$

being satisfied for all $x\in P$, $t\in[0,T]$, $\nu\in[0,\nu_0]$.

We will consider functions q_1^k, q_2^k defined by

$$\left(q_{i,t}^k\right)^2 := f_i^2-2t\delta\phi_i-\int_0^t a_i^k(s)v_i\cdot\left(\nabla h_{i,s}^2+2\nabla p[a_1^k(s)v_1,h_{1,s}]+2\nabla p[a_2^k(s)v_2,h_{2,s}]\right)\mathrm{d}s, \tag{2.71}$$

$i=1,2$, $k\in\mathbb{N}$, for some functions $a_1^k,a_2^k\in C^\infty(\mathbb{R};[-1,1])$ (which we shall call *oscillatory processes* and which we discuss below). Recall that we use the convention $q_{i,t}^k(\cdot)\equiv q_i^k(\cdot,t)$ (see (2.17)). We will show that, given a particular choice of the oscillatory processes a_1^k,a_2^k, the vector field

$$u(x,t):=u[a_1^k(t)v_1,q_{1,t}^k](x)+u[a_2^k(t)v_2,q_{2,t}^k](x), \tag{2.72}$$

is a solution to Proposition 2.11 for sufficiently large k. Note that such u is rotationally invariant (recall Section 2.2.3). Before proceeding to the proof, we comment on this strategy in an informal way.

Forget, for the moment, about the functions q_1^k, q_2^k, and let us try to attack Proposition 2.11 directly. We observe that part (ii) and the facts that $h_{1,t}$, $h_{2,t}$ have disjoint supports $\overline{U_1}$, $\overline{U_2}$ (respectively) and that $(v_1,h_{1,t},\phi_1)$, $(v_2,h_{2,t},\phi_2)$ are structures on U_1, U_2 (respectively) suggest looking at the velocity field of the form

$$\widetilde{u}(x,t):=u[v_1,h_{1,t}](x)+u[v_2,h_{2,t}](x). \tag{2.73}$$

In other words we have

$$|\widetilde{u}(x,0,t)|^2=h_{1,t}^2(x)+h_{2,t}^2(x)\qquad x\in P, t\in[0,T],$$

so that claim (ii) is satisfied in an exact sense (rather than in an approximate sense with accuracy θ). This might look promising, but, recalling the definition of h_1, h_2 (see (2.61), (2.62)) we see that

$$\partial_t\,|\widetilde{u}(x,0,t)|^2=-2\delta(\phi_1+\phi_2)(x)+v_2(x)\cdot F[v_1,h_{1,t}](x),$$

and at this point is is not clear how to relate the right-hand side to the terms

$$-u\cdot\nabla\left(|u|^2+2p\right)+\nu u\cdot\Delta u,$$

which are required by (2.70) (that is by (iii)). Thus the velocity field $\widetilde{u}$ seems unlikely to be a solution of Proposition 2.11. In order to proceed one needs to make use of two degrees of freedom available in the construction of $\widetilde{u}$. The first of them is the fact that claim (ii) of Proposition 2.11 only requires $|u(x,t)|$ to "keep close" to $h_t(R^{-1}x)$ as t varies between 0 and T (rather than to be equal to it), which we have already pointed out above. The second one is that $|\widetilde{u}(x,0,t)|$ is expressed only in terms of $h_{1,t}$, $h_{2,t}$. Thus a velocity field of the form

$$\overline{u}(x,t) \coloneqq u[a_1(t)v_2, h_{1,t}](x) + u[a_2(t)v_2, h_{2,t}](x)$$

has the same absolute value $|\overline{u}|$ as $|\widetilde{u}|$ for any choice of $a_1, a_2 \colon \mathbb{R} \to [-1,1]$. Recall also that since $|a_1|, |a_2| \le 1$,

$$(a_i(t)v_i, h_{i,t}, \phi_i) \qquad \text{is a structure on } U_i, i=1,2,$$

(recall (2.67)) and so $\overline{u}$ is well-defined. By introducing the functions q_1^k, q_2^k (in (2.71)) we make use of these two degrees of freedom.

We now proceed to a discussion of some elementary properties of these functions, and we show in Section 2.3.2 that considering them is a good idea; namely that (2.72) is a solution of the proposition for sufficiently large k.

First note that, as in the case of $h_{i,t}$, $q_{i,t}^k$ differs from f_i only on the compact set $\operatorname{supp}\phi_i$, $i=1,2$. Secondly,

$$\partial_t \left(q_{i,t}^k\right)^2 = -2\delta\phi_i - a_i^k(t)v_i \cdot \left(\nabla h_{i,t}^2 + 2\nabla p[a_1^k(t)v_1, h_{1,t}] + 2\nabla p[a_2^k(t)v_2, h_{2,t}]\right). \quad (2.74)$$

Finally, we will show in Section 2.3.3 that, given a particular choice of the oscillatory processes $a_1^k, a_2^k \in C^\infty(\mathbb{R},[-1,1])$ (which are a part of the definition of q_1^k, q_2^k, recall (2.71)),

$$\begin{cases} q_{i,t}^k \to h_{i,t} \\ \text{and} \\ D^l q_{i,t}^k \to D^l h_{i,t} \end{cases} \qquad \text{uniformly in } P\times[0,T], i=1,2, \text{ for each } l\ge 1 \quad (2.75)$$

as $k\to\infty$. Recalling properties of the functions $h_{1,t}$, $h_{2,t}$ (see Lemma 2.10), we see that this convergence gives in particular that for sufficiently large k

$$q_{i,t}^k > |v_i| \qquad \text{in } \operatorname{supp}\phi_i \text{ for } t\in[0,T], i=1,2,$$

and so by continuity (as in the proof of Lemma 2.10)

$$q_{i,t}^k > |v_i| \geq 0 \qquad \text{in } U_i \text{ for } t \in (-\delta_k, T+\delta_k), \tag{2.76}$$

for some $\delta_k \in (0,\delta)$. Thus for sufficiently large k

$$(v_i, q_{i,t}^k, \phi_i) \text{ is a structure on } U_i \qquad \text{for } t \in (-\delta_k, T+\delta_k), i=1,2,$$

and thus, since $|a_1^k|, |a_2^k| \in [-1,1]$, also

$$(a_i^k(t) v_i, q_{i,t}^k, \phi_i) \text{ is a structure on } U_i \qquad \text{for } t \in (-\delta_k, T+\delta_k), i=1,2. \tag{2.77}$$

Moreover, (2.76) and the fact that all terms on the right-hand side of (2.71) are smooth (recall (2.36) for the smoothness of the pressure) give

$$q_i^k \in C^\infty(P \times (-\delta_k, T+\delta_k); [0,\infty)), \qquad i=1,2.$$

2.3.2 The proof of the claims of the proposition

Using the above properties of the functions q_1^k, q_2^k, we now show that for k sufficiently large the vector field u given by (2.72),

$$u(x,t) \coloneqq u[a_1^k(t)v_1, q_{1,t}^k](x) + u[a_2^k(t)v_2, q_{2,t}^k](x),$$

with $\eta \coloneqq \delta_k$ satisfies the claims of Proposition 2.11.

Claim (i) and the smoothness of u on $\mathbb{R}^3 \times (-\eta, T+\eta)$ follow directly from (2.77), the smoothness of the oscillatory processes a_1^k, a_2^k on $\mathbb{R}$ (which we are about to construct in the next section) and from the smoothness of q_1^k, q_2^k stated above.

Claim (ii) is equivalent to (2.69) (due to rotational invariance of u), and thus its first part follows by writing

$$|u(x,0,0)| = q_{1,0}^k(x) + q_{2,0}^k(x) = f_1(x) + f_2(x) = h_0(x).$$

The second part follows directly from the convergence (2.75) by taking k sufficiently large such that

$$\left|(q_{1,t}^k + q_{2,t}^k)^2 - h_t^2\right| < \theta \qquad \text{in } P, t \in [0,T].$$

For such k we obtain

$$\left| |u(x,0,t)|^2 - h_t(x)^2 \right| = \left| (q_{1,t}^k(x) + q_{2,t}^k(x))^2 - h_t(x)^2 \right| < \theta,$$

as required.

As for Claim (iii), first recall that $p(t)$, the pressure function corresponding to $u(t)$, is (due to (2.34)) given by

$$p(t) = p^*[a_1^k(t)v_1, q_{1,t}^k] + p^*[a_2^k(t)v_2, q_{2,t}^k],$$

and so in particular

$$p(x,0,t) = p[a_1^k(t)v_1, q_{1,t}^k](x) + p[a_2^k(t)v_2, q_{2,t}^k](x).$$

Recalling that Claim (iii) is equivalent to (2.70), that is the Navier–Stokes inequality restricted to P,

$$\partial_t |u(x,0,t)|^2 \leq -u(x,0,t)\cdot \nabla \left(|u(x,0,t)|^2 + 2p(x,0,t)\right) + 2\nu\, u(x,0,t)\cdot \Delta u(x,0,t),$$

where $\nu \in [0,\nu_0]$ (recall (2.68) for the choice of ν_0), we fix $x\in P$, $t\in[0,T]$ and we consider two cases.

Case 1. $\phi_1(x)+\phi_2(x) < 1$. For such x we have $v_1(x) = v_2(x) = 0$ and the Navier–Stokes inequality follows trivially for all k by writing

$$\begin{aligned}\partial_t |u(x,0,t)|^2 &= \partial_t q_{1,t}^k(x)^2 + \partial_t q_{2,t}^k(x)^2 \\ &= -2\delta(\phi_1(x)+\phi_2(x)) \\ &\leq 0 \\ &\leq -u(x,0,t)\cdot \nabla \left(|u(x,0,t)|^2 + 2p(x,0,t)\right) + 2\nu\, u(x,0,t)\cdot \Delta u(x,0,t),\end{aligned} \tag{2.78}$$

where we used (2.43) and (2.44) in the last step.

Case 2. $\phi_1(x)+\phi_2(x) = 1$. In this case we need to use the convergence (2.75) to take k sufficiently large such that

$$|v_i| \left(\left| \nabla (q_{i,t}^k)^2 - \nabla h_{i,t}^2 \right| + 2 \sum_{j=1,2} \left| \nabla p[a_j^k(t)v_j, q_{j,t}^k] - \nabla p[a_j^k(t)v_j, h_{j,t}] \right| \right) \leq \delta/2 \tag{2.79}$$

in P and

$$\begin{aligned}&\nu_0 \left| u[a_i^k(t)v_i, q_{i,t}^k]\cdot \Delta u[a_i^k(t)v_i, q_{i,t}^k] \right| \\ &\leq \nu_0 \left| u[a_i^k(t)v_i, h_{i,t}]\cdot \Delta u[a_i^k(t)v_i, h_{i,t}] \right| + \delta/8 \leq \delta/4\end{aligned} \tag{2.80}$$

in $\mathbb{R}^3$, for $t \in [0,T]$, $i = 1,2$ (the last inequality follows from the definition of ν_0, see (2.68)). We obtain

$$\begin{aligned}
\partial_t |u(x,0,t)|^2 &= \partial_t q_{1,t}^k(x)^2 + \partial_t q_{2,t}^k(x)^2 \\
&= -2\delta - \left(a_1^k(t) v_1(x) + a_2^k(t) v_2(x)\right) \cdot \nabla \left(h_{1,t}(x)^2 + h_{2,t}(x)^2\right. \\
&\qquad\qquad \left. + 2p[a_1^k(t)v_1, h_{1,t}](x) + 2p[a_2^k(t)v_2, h_{2,t}](x)\right) \\
&\leq -\delta - \left(a_1^k(t) v_1(x) + a_2^k(t) v_2(x)\right) \cdot \nabla \left(q_{1,t}^k(x)^2 + q_{2,t}^k(x)^2\right. \\
&\qquad\qquad \left. + 2p[a_1^k(t)v_1, q_{1,t}^k](x) + 2p[a_2^k(t)v_2, q_{2,t}^k](x)\right) \\
&= -\delta - u_1(x,0,t)\partial_{x_1}\left(|u(x,0,t)|^2 + 2p(x,0,t)\right) \\
&\qquad - u_2(x,0,t)\partial_{x_2}\left(|u(x,0,t)|^2 + 2p(x,0,t)\right),
\end{aligned} \tag{2.81}$$

and so, recalling that $\partial_{x_3}|u(x,0,t)|^2 = \partial_{x_3} p(x,0,t) = 0$ (as a property of rotationally invariant functions, see (2.32) and (2.37)),

$$\begin{aligned}
\partial_t |u(x,0,t)|^2 &\leq -\delta - u(x,0,t) \cdot \nabla \left(|u(x,0,t)|^2 + 2p(x,0,t)\right) \\
&\leq 2\nu\, u(x,0,t) \cdot \Delta u(x,0,t) - u(x,0,t) \cdot \nabla \left(|u(x,0,t)|^2 + 2p(x,0,t)\right)
\end{aligned}$$

for all $\nu \in [0,\nu_0]$, where we used (2.80) in the last step.

Thus we have shown that for sufficiently large k the Navier–Stokes inequality (2.70) holds for all $x \in P$, $t \in [0,T]$ and $\nu \in [0,\nu_0]$, which gives (iii), as required.

2.3.3 The oscillatory processes

Here we construct the oscillatory processes $a_1^k, a_2^k \in C^\infty(\mathbb{R},[-1,1])$, $k \geq 1$, such that the functions q_1^k, q_2^k (given by (2.71)) converge to h_1, h_2 (respectively) as in (2.75). As outlined in Section 2.3.1 this completes the proof of Theorem 2.1 given the *geometric arrangement* (which we construct in Section 2.4).

As for the strategy for choosing a_1^k, a_2^k we will divide $[0,T]$ into $4k$ subintervals and on each those subintervals we will let each of a_1^k, a_2^k equal 1, -1 or 0 (except for a set of times of measure less than $1/k$) in a particular configuration. The configuration is such that the resulting $q_{1,t}^k$, $q_{2,t}^k$ oscillate near $h_{1,t}$, $h_{2,t}$ as t varies between 0 and T, and such that the oscillations grow in frequency (that is the number of subintervals increases with k) and decrease in magnitude (that is we obtain convergence (2.75)), see Fig. 2.6 for a sketch.

We employ this strategy in the proof of the theorem below.

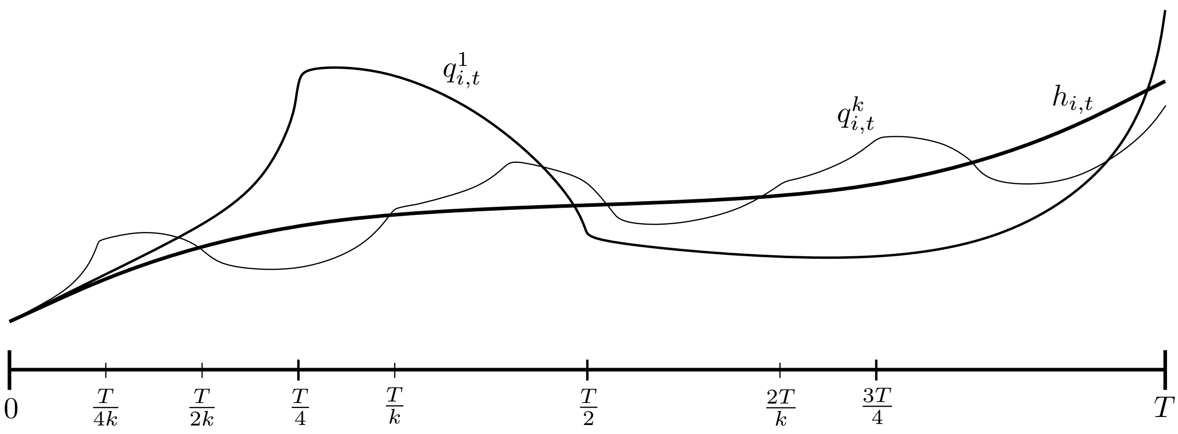

Figure 2.6: The strategy for the choice of a_i^k, $i=1,2$. This sketch illustrates how the choice of a_i^k's causes $q_{i,t}^k$'s to "oscillate around $h_{i,t}$" as t varies between 0 to T. Here $k=3$.

Theorem 2.12 (existence of the oscillatory processes)**.** *For each $k\geq 1$ there exist a pair of functions $a_i^k\in C^\infty(\mathbb{R};[-1,1])$, $i=1,2$, such that*

$$\begin{aligned}&\int_0^t a_i^k(s)\left(G_i(x,s)+F_{i,1}\left(x,s,a_1^k(s)\right)+F_{i,2}\left(x,s,a_2^k(s)\right)\right)\mathrm{d}s\\ &\qquad\xrightarrow{k\to\infty}\begin{cases}\frac12\int_0^t\left(F_{2,1}(x,s,1)-F_{2,1}(x,s,0)\right)\mathrm{d}s & i=2,\\ 0 & i=1\end{cases}\end{aligned}\tag{2.82}$$

uniformly in $(x,t)\in P\times[0,T]$ for any bounded and uniformly continuous functions

$$G_i\colon P\times[0,T]\to\mathbb{R},\qquad F_{i,l}\colon P\times[0,T]\times[-1,1]\to\mathbb{R},$$

$i,l=1,2$, satisfying

$$F_{i,l}(x,t,-1)=F_{i,l}(x,t,1)\qquad \text{for } x\in P, t\in[0,T], i,l=1,2.$$

Note that this theorem gives (2.75) simply by taking

$$\begin{aligned}G_i(x,t)&\coloneqq v_i(x)\cdot\nabla h_i(x,t)^2,\\ F_{i,l}(x,t,a)&\coloneqq 2v_i(x)\cdot\nabla p[av_l,h_{l,t}](x)\end{aligned}$$

(recall $p[v,f]=p[-v,f]$ by Lemma 2.6 (i)), and so such $F_{i,l}$'s satisfy the requirement $F_{i,l}(x,t,-1)=F_{i,l}(x,t,1)$ above) and by taking

$$\begin{aligned}G_i(x,t)&\coloneqq D^\alpha\left(v_i(x)\cdot\nabla h_i(x,t)^2\right),\\ F_{i,l}(x,t,a)&\coloneqq D^\alpha\left(2v_i(x)\cdot\nabla p[av_l,h_{l,t}](x)\right)\end{aligned}$$

for any given multiindex $\alpha = (\alpha_1, \alpha_2)$.

Before proceeding to the proof of Theorem 2.12 we pause for a moment to comment on the meaning of the theorem and the convergence (2.75) in an informal manner. Recall that (2.71) includes terms of the form

$$2\int_0^t a_i^k(s) v_i \cdot \nabla p[a_l^k(s) v_l, h_{l,s}] \mathrm{d}s, \quad i,l \in \{1,2\}.$$

Note that each of such terms represent, in a sense, an influence of the set U_l (together with the structure $(a_l^k(s)v_l, h_{l,s}, \phi_l)$) on the set U_i; namely it vanishes outside U_i and it uses the nonlocal character of the pressure function $p[\cdot,\cdot]$ (that is the fact that the pressure function $p[a_l^k(s)v_l, h_{l,s}]$ does not vanish on U_i). Thus we see from (2.82) that the role of the oscillatory processes a_1^k, a_2^k is to "select" only the influence of U_1 on U_2 as $k\to\infty$ (except for this the oscillatory behaviour of the processes makes the terms $\int_0^t a_i^k(s) v_i \cdot \nabla h_{i,s}^2$, $i=1,2$, vanish as $k\to\infty$). Note this is the expected behaviour since we want to show the convergence (2.75) and of the two functions h_1, h_2 only h_2 includes an influence from U_1 (recall (2.61), (2.62)). The construction of such oscillatory processes is clear from the following auxiliary considerations, in which we forget, for a moment, about the smoothness requirement.

Let $f\colon [-1,1]\to\mathbb{R}$ be such that $f(-1)=f(1)$ and let functions $b_1, b_2\colon [0,T]\to[-1,1]$ be such that

$$b_1(t) = \begin{cases} 1 & t\in(0,T/4),\\ -1 & t\in(T/4,T/2),\\ 0 & t\in(T/2,T),\end{cases} \qquad b_2(t) = \begin{cases} 1 & t\in(0,T/2),\\ -1 & t\in(T/2,T).\end{cases} \tag{2.83}$$

Then

$$\int_0^T b_i(s) f\left(b_l(s)\right) \mathrm{d}s = \begin{cases} \frac{T}{2}(f(1)-f(0)) & (i,l)=(2,1),\\ 0 & (i,l)\neq(2,1),\end{cases} \tag{2.84}$$

that is the choice of b_1, b_2 is such that they "pick" the value $T(f(1)-f(0))/2$ only for the choice of indices $(i,l)=(2,1)$. Clearly, given $(i_0,l_0)\in\{1,2\}^2$ one could choose b_1, b_2 that pick this value only for the choice of indices $(i,l)=(i_0,l_0)$.

More generally, let f be also a function of time, $f\colon[0,T]\times[-1,1]\to\mathbb{R}$ with $f(t,-1)=f(t,1)$ for all t such that f is almost constant with respect to the first variable, i.e. for some $\epsilon>0$

$$\sup_{t\in[0,T]} f(t,a) - \inf_{t\in[0,T]} f(t,a) < \epsilon, \qquad a\in[-1,1].$$

Then

$$\int_0^T b_i(t) f\left(s, b_l(s)\right) \mathrm{d}s = \begin{cases} \frac{1}{2}\int_0^T \left(f(s,1)-f(s,0)\right)\mathrm{d}s + T\,O(\epsilon) & (i,l)=(2,1), \\ T\,O(\epsilon) & (i,l)\neq(2,1). \end{cases}$$

These observations are helpful in finding $b_1^k, b_2^k \colon [0,T] \to [-1,1]$ such that for every continuous f

$$\int_0^T b_i^k(s) f\left(s, b_l^k(s)\right) \mathrm{d}s \to \begin{cases} \frac{1}{2}\int_0^T (f(s,1)-f(s,0))\mathrm{d}s & (i,l)=(2,1), \\ 0 & (i,l)\neq(2,1) \end{cases} \tag{2.85}$$

as $k\to\infty$. Indeed one can take b_1^k, b_2^k to be oscillations of the form (2.83), but of higher frequency,

$$b_1^k(t) := b_1(kt), \qquad b_2^k(t) := b_2(kt), \tag{2.86}$$

where we extended b_1, b_2 T-periodically to the whole line, see Fig. 2.7.

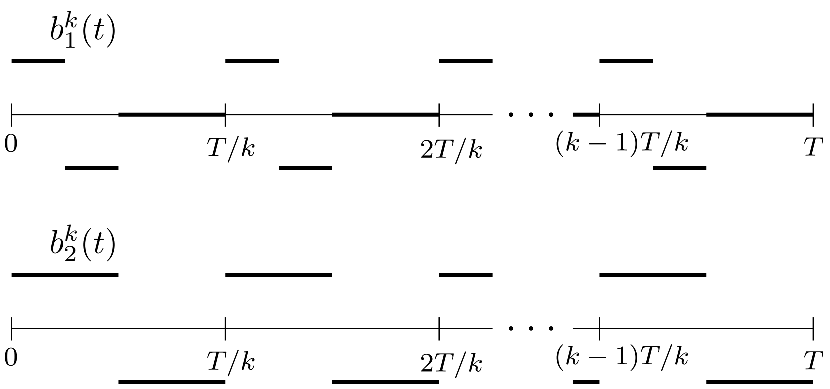

Figure 2.7: The functions b_1^k, b_2^k.

In order to see that such a choice gives the convergence in (2.85) note that continuity of f implies that $\varepsilon_k\to 0$ as $k\to\infty$, where $\varepsilon_k>0$ is the smallest positive number such that

$$|f(t,a)-f(s,a)| \leq \varepsilon_k \tag{2.87}$$

whenever $a\in[-1,1]$ and $s,t\in[0,T]$ are such that $|t-s|\leq T/k$. Thus, if $(i,l)=(2,1)$

we write

$$
\begin{aligned}
\int_0^T b_2^k(s) f\left(s, b_1^k(s)\right) \mathrm{d}s &= \sum_{p=0}^{k-1} \int_{pT/k}^{(p+1)T/k} b_2^k(s) f\left(s, b_1^k(s)\right) \mathrm{d}s \\
&= \sum_{p=0}^{k-1} \left(\int_{pT/k}^{(p+1/2)T/k} f(s,1)\, \mathrm{d}s - \int_{(p+1/2)T/k}^{(p+1)T/k} f(s,0)\, \mathrm{d}s \right) \\
&= \sum_{p=0}^{k-1} \left(\frac{1}{2} \int_{pT/k}^{(p+1)T/k} f(s,1)\, \mathrm{d}s - \frac{1}{2} \int_{pT/k}^{(p+1)T/k} f(s,0)\, \mathrm{d}s + \frac{T}{k} O(\varepsilon_k) \right) \\
&= \frac{1}{2} \int_0^T (f(s,1) - f(s,0)) \mathrm{d}s + T\, O(\varepsilon_k).
\end{aligned}
\tag{2.88}
$$

Thus

$$
\int_0^T b_2^k(s) f\left(s, b_1^k(s)\right) \mathrm{d}s \to \frac{1}{2} \int_0^T (f(s,1) - f(s,0)) \mathrm{d}s \qquad \text{as } k \to \infty,
$$

and in the same way one can show that

$$
\int_0^T b_i^k(s) f\left(s, b_l^k(s)\right) \mathrm{d}s \to 0 \qquad \text{as } k \to \infty
$$

if $(i,l) \neq (2,1)$. Therefore we obtain (2.85).

In a similar way one can show that for such choice of b_1^k, b_2^k, the upper limit of the integrals in (2.85) can be replaced by any $t \in [0,T]$, that is

$$
\int_0^t b_i^k(s) f\left(s, b_j^k(s)\right) \mathrm{d}s \to \begin{cases} \frac{1}{2} \int_0^t (f(s,1) - f(s,0)) \mathrm{d}s & (i,l) = (2,1), \\ 0 & (i,l) \neq (2,1) \end{cases}
\tag{2.89}
$$

as $k \to \infty$, uniformly in $t \in [0,T]$. To this end, given $t \in [0,T]$ let $q \in \{0, \ldots, k-1\}$ be such that $t \in [qT/k, (q+1)T/k)$ and write the left-hand side of (2.89) above as

$$
\sum_{p=0}^{q-1} \int_{pT/k}^{(p+1)T/k} b_2^k(s) f(s, b_1^k(s)) \mathrm{d}s + \int_{qT/k}^{t} b_2^k(s) f(s, b_1^k(s)) \mathrm{d}s.
$$

The sum from $p=0$ to $q-1$ can be treated in the same way as the sum over all p's in the calculation (2.88) above to give

$$
\frac{1}{2} \int_0^{qT/k} (f(s,1) - f(s,0)) \mathrm{d}s + \frac{qT}{k} O(\varepsilon_k).
$$

The remaining term can be treated using boundedness of f (note $|f| \leq N$ for some $N > 0$ due to continuity of f and to the fact that its domain $[0,T] \times [-1,1]$ is compact) by writing

$$
\left| \int_{qT/k}^{t} b_2^k(s) f(s, b_1^k(s)) \mathrm{d}s - \frac{1}{2} \int_{qT/k}^{t} (f(s,1) - f(s,0))\, \mathrm{d}s \right| \leq 2N\, |t - qT/k| \leq 2NT/k,
$$

and thus we obtain (2.89) in the case $(i,l)=(2,1)$. The case $(i,l)\neq(2,1)$ follows similarly.

Moreover, due to the oscillatory behaviour of b_1^k, b_2^k as k increases we also see that each of b_1^k, b_2^k converges to 0 in a weak sense, that is

$$\int_0^t b_i^k(s)g(s)\,\mathrm{d}s \to 0 \qquad \text{as } k\to\infty, i=1,2, \text{ uniformly in } t\in[0,T] \tag{2.90}$$

for any continuous $g\colon[0,T]\to\mathbb{R}$.

The above ideas are a basis of the proof of Theorem 2.12, in which x plays no role and the processes a_1^k, a_2^k are obtained by a smooth approximation of b_1^k, b_2^k, respectively.

Proof of Theorem 2.12. Let $b_1^k, b_2^k\colon[0,T]\to[-1,1]$ be defined by (2.86) above. Given $k\geq 0$ let $\varepsilon_k>0$ be the smallest number such that

$$|F_{i,l}(x,t,a)-F_{i,l}(x,s,a)|, |G_i(x,t)-G_i(x,s)|\leq\varepsilon_k, \qquad i,l=1,2 \tag{2.91}$$

whenever $x\in P$, $a\in[-1,1]$ and $t,s\in[0,T]$ are such that $|t-s|\leq T/k$. Due to the uniform continuity of $F_{i,l}$'s and G_i's we obtain $\varepsilon_k\to 0$ as $k\to\infty$. Moreover, from boundedness we obtain $N>0$ such that $|F_{i,l}|,|G_i|\leq N$ for $i,j=1,2$. Thus applying (2.89), with $f(t,a)\coloneqq F_{i,l}(x,t,a)$ (for every x) and with the continuity property (2.87) replaced by the uniform continuity of $F_{i,j}$'s (2.91) and by the boundedness $|F_{i,l}|\leq N$ we obtain

$$\int_0^t b_i^k(s)F_{i,l}\left(x,s,b_l^k(s)\right)\mathrm{d}s \to \begin{cases} \frac12\int_0^t(F_{2,1}(x,s,1)-F_{2,1}(x,s,0))\mathrm{d}s & (i,l)=(2,1),\\ 0 & (i,l)\neq(2,1)\end{cases}$$

as $k\to\infty$ uniformly in $x\in P$, $t\in[0,T]$. Similarly applying (2.90) with $g(t)\coloneqq G_i(x,t)$ we obtain

$$\int_0^t b_i^k(s)G_i(x,s)\,\mathrm{d}s\to 0 \qquad \text{as } k\to\infty$$

uniformly in $x\in P$, $t\in[0,T]$, $i=1,2$. Thus, altogether

$$\begin{aligned}&\int_0^t b_i^k(s)\left(G_i(x,s)+F_{i,1}\left(x,s,b_1^k(s)\right)+F_{i,2}\left(x,s,b_2^k(s)\right)\right)\mathrm{d}s\\ &\qquad\xrightarrow{k\to\infty}\begin{cases}\frac12\int_0^t\left(F_{2,1}(x,s,1)-F_{2,1}(x,s,0)\right)\mathrm{d}s & i=2,\\ 0 & i=1\end{cases}\end{aligned} \tag{2.92}$$

uniformly in $(x,t)\in P\times[0,T]$. Thus the oscillatory processes b_1^k, b_2^k (defined by (2.86)) satisfy all the claims of the theorem, except for the C^∞ regularity. To this end let

$a_1^k, a_2^k \in C^\infty(\mathbb{R}; [-1,1])$ be such that

$$\left|\left\{t \in [0,T] \colon a_i^k(t) \neq b_i^k(t)\right\}\right| \leq \frac{1}{k}, \qquad i = 1,2.$$

Such a_1^k, a_2^k can be obtained by extending b_1^k, b_2^k to the whole line by zero and mollifying. Clearly, such definition of the processes a_1^k, a_2^k and the boundedness $|F_{i,l}|, |G_i| \leq N$ gives that the difference between the left-hand sides of (2.82) and (2.92) is bounded by

$$6N/k \to 0 \qquad \text{as } k \to \infty,$$

which shows that these left-hand sides converge to the same limit

$$\begin{cases} \frac{1}{2}\int_0^t \left(F_{2,1}(x,s,1) - F_{2,1}(x,s,0)\right) \mathrm{d}s & i = 2, \\ 0 & i = 1 \end{cases}$$

uniformly in $(x,t) \in P \times [0,T]$, as required. $\square$

2.4 The geometric arrangement

In this section we construct the *geometric arrangement*, that is $T > 0$, $\tau \in (0,1)$, $z \in \mathbb{R}^3$, sets $U_1, U_2 \Subset P$ with disjoint closures and the respective structures (v_1, f_1, ϕ_1), (v_2, f_2, ϕ_2) such that

$$f_2^2 + T v_2 \cdot F[v_1, f_1] > |v_2|^2 \quad \text{in } U_2,$$

$$f_2^2(y) + T v_2(y) \cdot F[v_1, f_1](y) > \tau^{-2}\left(f_1(R^{-1}x) + f_2(R^{-1}x)\right)^2$$

for all $x \in G = R(\overline{U_1} \cup \overline{U_2})$, where $y = R^{-1}(\Gamma(x))$, and that

$$\Gamma(G) \subset G.$$

According to the considerations of Section 2.3, this construction concludes the proof of Theorem 2.1.

Let

$$U \coloneqq (-1,1) \times (1/8, 7/8),$$

and let $v \in C_0^\infty(U; \mathbb{R}^2)$ be any vector field satisfying

$$\begin{cases} v_1(-x_1, x_2) = v_1(x_1, x_2), \\ v_2(-x_1, x_2) = -v_2(x_1, x_2) \end{cases}$$

and

$$\operatorname{div}(x_2\, v(x_1,x_2) = 0, \qquad (x_1,x_2) \in P.$$

One can take for instance

$$v(x_1,x_2) \coloneqq x_2^{-1} J \left((-(x_2-1/2), x_1) I_{A((0,1/2),1/8,1/16)}(x_1,x_2) \right),$$

where J denotes a sufficiently fine mollification, as in the recipe for a structure presented in Section 2.2.5. Following the recipe, let $f \in C_0^\infty(P;[0,\infty))$ be such that $\operatorname{supp} f = \overline{U}$, $f > |v|$ in U and $Lf > 0$ at points of U of sufficiently small distance from ∂U. Furthermore, construct f in a way that

$$f(-x_1,x_2) = f(x_1,x_2).$$

We show existence of such f in Lemma 2.18. Let $\phi \in C_0^\infty(U;[0,1])$ be a cutoff function such that $\operatorname{supp} v \subset \{\phi = 1\}$ and $Lf > 0$ in $U \setminus \{\phi = 1\}$. Thus we obtained a structure (v,f,ϕ) on U. Consider the pressure interaction function $F \coloneqq F[v,f]$ (recall (2.47)) and let $A \in \mathbb{R}$, $B,C,D,N > 0$ and $\kappa = 10^4 C/D$ be the constants given by Lemma 2.9.

Since the structure (v,f,ϕ) satisfies the condition of Lemma 2.6 (ii), we see that the first component of $F[v,f]$ is odd when restricted to the x_1 axis, that is

$$F_1(-x_1,0) = -F_1(x_1,0), \quad x_1 \in \mathbb{R}.$$

Thus, in the view of Lemma 2.9 (ii), we observe that $A \neq 0$ and

$$-B = F_1(-A,0) = \min_{x_1\in\mathbb{R}} F_1(x_1,0).$$

2.4.1 A simplified geometric arrangement

At this point we pause for the moment to present a certain simplified geometric arrangement. Although the simplified arrangement has the unfortunate property of being impossible, it offers a good perspective on the main difficulty of geometric arrangement. We also explain the strategy for overcoming this difficulty. The reader who is not interested in the simplified arrangement is referred to the next section (Section 2.4.2), where we proceed with the presentation of the geometric arrangement proper.

From Lemma 2.9 (ii) we see that there exists a rectangle $U_2 \Subset P$ such that $F_1[v,f] \geq B/2$ in U_2. Let $v_2 = (v_{21}, v_{22}) \in C_0^\infty(U_2;\mathbb{R}^2)$ be such that $\operatorname{div}(x_2 v_2(x_1,x_2)) = 0$ for $(x_1,x_2) \in P$,

$$v_{22} = 0,\ v_{21} \geq 0, \text{ and } v_2 = (1,0) \text{ in some closed rectangle } K \subset U_2.$$

Warning 2.13. *Such v_2 does not exist!*
Indeed, take $w := x_2 v_2$ and let K' be a rectangle such that its left edge is the left edge of K and its right edge lies on ∂U_2. Integrating $\operatorname{div} w$ *over K we obtain*

$$0 = \int_K \operatorname{div} w = \int_{\partial K'} w \cdot n = \int_{\partial_L K'} w_1 = \int_{\partial_L K'} x_2 > 0,$$

where $\partial_L K'$ denotes the left edge of K'.

Let (z_1, z_2) be an interior point of K, $z := (z_1, z_2, 0) \in \mathbb{R}^3$, $U_1 := U$, $v_1 := v$, $f_1 := f$, $\phi_1 := \phi$ (note then $F = F[v_1, f_1]$) and let $\tau \in (0,1)$ be sufficiently small such that

$$R^{-1}\left(\tau R(\overline{U_1} \cup \overline{U_2}) + z\right) \subset K, \tag{2.93}$$

see Fig. 2.8. Let f_2, ϕ_2 be any functions such that (v_2, f_2, ϕ_2) is a structure on U_2 (that

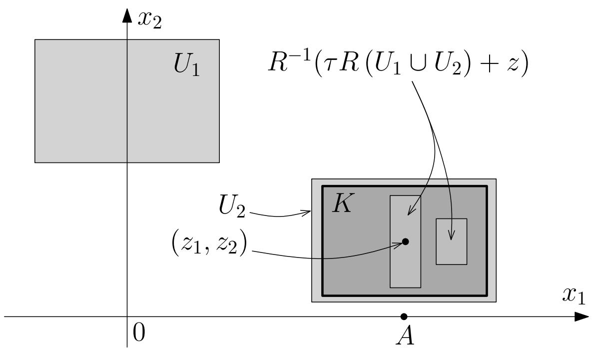

Figure 2.8: The simplified arrangement. Note that it is not quite correct, see Warning 2.13.

is define f_2, ϕ as described in the recipe in Section 2.2.5). Then (2.50) follows trivially for every $T > 0$ by noting that

$$v_2 \cdot F = v_{21} F_1 \geq 0 \quad \text{ in } U_2, \tag{2.94}$$

and so

$$f_2^2 + T v_2 \cdot F = f_2^2 + T v_{21} F_1 \geq f_2^2 > |v_2|^2 \qquad \text{in } U_2. \tag{2.95}$$

Moreover, (2.51) follows provided we choose $T > 2\tau^{-2} \|f_1 + f_2\|_\infty^2 / B$. Indeed, then we obtain

$$T F_1 \geq \tau^{-2} \|f_1 + f_2\|_\infty^2 \quad \text{ in } U_2, \tag{2.96}$$

and so letting $x \in R(\overline{U_1} \cup \overline{U_2})$ and $y := R^{-1}(\tau x + z)$ we see that (2.93) gives $y \in K$ and thus

$$f_2^2(y) + T v_2(y) \cdot F(y) = f_2^2(y) + T F_1(y) \geq \tau^{-2} \|f_1 + f_2\|_\infty^2, \tag{2.97}$$

as required.

This concludes the simplified geometric arrangement. Note, however, it does not exist due to Warning 2.13. In fact, it is clear that v_2 cannot have $(1,0)$ as the only direction, which is, roughly speaking, a consequence of the fact that any weakly divergence-free vector field in $\mathbb{R}^2$ must "run in a loop", cf. Fig 2.3. Thus, for each of the quantities

$$F_1, F_2, -F_1, -F_2$$

there exists a region in P such that at least one of the ingredients of the inner product

$$v_2 \cdot F = v_{21} F_1 + v_{22} F_2$$

gives a quantity with the magnitude v_{21} or v_{22} (the size of the magnitude obviously depending on the choice of v_2). Thus the calculations (2.95), (2.97), in which we used the very convenient properties (2.94), (2.96) immediately become useless and at this point it is not clear how to estimate $v_2 \cdot F$ to obtain the required relations (2.50), (2.51).

In the remainder of this section we sketch a more elaborate construction of sets U_1 and U_2 as well as their structures that solve this difficulty. In particular we point out the relations that will replace (2.94), (2.96) in showing the required relations (2.50), (2.51). The construction is then presented in detail in the following Sections 2.4.2-2.4.5.

First of all, we will consider the rescaling of the set U and its structure (v, f, ϕ), that is for $\alpha \in \mathbb{R}, \rho > 0$ and $\sigma > 0$ we will consider a set $U^{\alpha,\rho}$ and a structure $(v^{\alpha,\rho,\sigma}, f^{\alpha,\rho,\sigma}, \phi^{\alpha,\rho})$ on $U^{\alpha,\rho}$. Here α corresponds to a translation in the x_1 direction, ρ scales the size of U and σ scales the magnitude of v and f. We will observe that manipulating the values of α, ρ, σ gives us certain amount of freedom in the manipulation of the shape of the pressure interaction function

$$F^{\alpha,\rho,\sigma} := F[v^{\alpha,\rho,\sigma}, f^{\alpha,\rho,\sigma}],$$

and so we will consider a disjoint union of U together with its two rescalings,

$$U \cup U^{a',r'} \cup U^{a'',r''},$$

along with the corresponding structure

$$(v, f, \phi) + \left(v^{a',r',s'}, f^{a',r',s'}, \phi^{a',r'} \right) + \left(v^{a'',r'',s''}, f^{a'',r'',s''}, \phi^{a'',r''} \right),$$

where the sum is understood in an entry-wise sense. Here, the values of $a', a'', r', r'', s', s''$ will be chosen in a particular way, roughly speaking such that the (joint) pressure interaction function

$$H := F + F^{a',r',s'} + F^{a'',r'',s''}$$

enjoys a similar decay to F (recall Lemma 2.9 (iii)) and, when restricted to the x_1 axis, its first component H_1 admits maximum $7B$ at A and minimum greater than or equal to $-1.005B$ (rather than maximum B and minimum $-B$, which is the case for F_1), see Fig. 2.9. Then, given a small parameter $\varepsilon > 0$ we will find numbers $d, r = O(1/\varepsilon)$

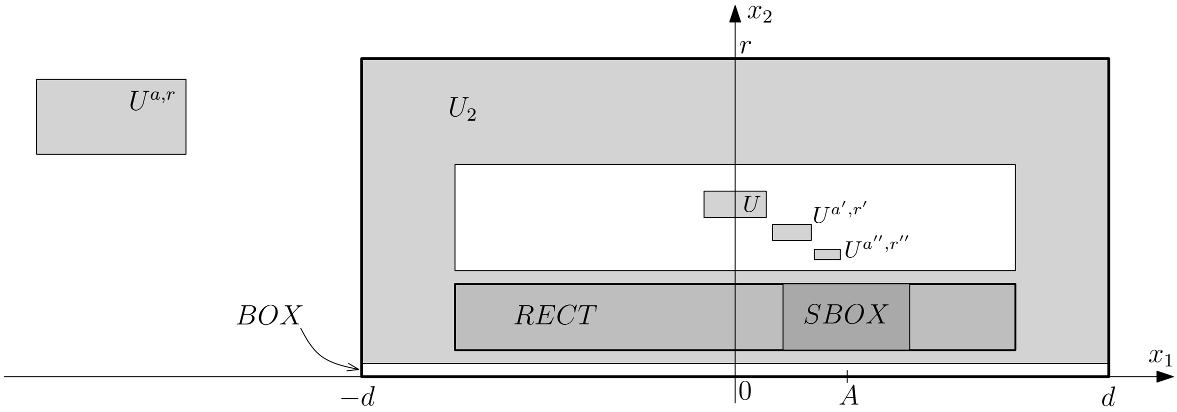

Figure 2.9: A sketch of the geometric arrangement (see Fig. 2.14 for a more detailed sketch). Some proportions are not conserved on the sketch.

with $d >> r$, $U_2 \Subset P$ and $v_2 \in C_0^\infty(U_2; \mathbb{R}^2)$ such that

$$U_2 \subset BOX := [-d, d] \times [0, r],$$

$$U_2 \text{ is a rectangular ring encompassing } U \cup U^{a',r'} \cup U^{a'',r''},$$

namely $U_2 = V \setminus \overline{W}$ where $V, W \Subset P$ are rectangles such that

$$U \cup U^{a',r'} \cup U^{a'',r''} \Subset W \Subset V,$$

see Fig. 2.9, and

$$v_2 = (1, 0) \quad \text{in } RECT \subset U_2,$$

where $RECT$ will be a carefully chosen rectangle located sufficiently close to the x_1 axis so that

$$\begin{aligned} &H_1 \geq -1.01B \text{ in } RECT, \\ &H_1 \geq 6.99B \text{ in some rectangle } SBOX \subset RECT. \end{aligned} \tag{2.98}$$

We will then choose τ, z such that

$$R^{-1}(\tau R(BOX) + z) \subset SBOX, \tag{2.99}$$

see Fig. 2.14, and we will define a pair of numbers $a = O(-\varepsilon^{-2})$, $s = O(\varepsilon^{-5/2})$ such that the rescaling $U^{a,r}$ of U together with the rescaled structure $(v^{a,r,s}, f^{a,r,s}, \phi^{a,r})$ satisfies

$$R^{-1}\left(\tau R\left(\overline{U^{a,r}}\right) + z\right) \subset RECT, \tag{2.100}$$

see Fig. 2.14, and that the pressure interaction function $F^{a,r,s} = F[v^{a,r,s}, f^{a,r,s}]$ is of particular size when restricted to BOX, that is $F_2^{a,r,s}$ is small (in some sense) and

$$1.03B \leq F_1^{a,r,s} \leq 1.05B \qquad \text{in } BOX. \tag{2.101}$$

For this we will crucially need the last property in Lemma 2.9, which, roughly speaking, quantifies the decay (in x_1) of the pressure interaction function in a precise way. We will then set

$$U_1 \coloneqq U \cup U^{a',r'} \cup U^{a'',r''} \cup U^{a,r}$$

together with the structure

$$\begin{aligned}(v_1, f_1, \phi_1) \coloneqq (v, f, \phi) &+ \left(v^{a',r',s'}, f^{a',r',s'}, \phi^{a',r'}\right) \\ &+ \left(v^{a'',r'',s''}, f^{a'',r'',s''}, \phi^{a'',r''}\right) + \left(v^{a,r,s}, f^{a,r,s}, \phi^{a,r}\right),\end{aligned}$$

so that the (total) pressure interaction function is

$$F^* \coloneqq F[v_1, f_1] = H + F^{a,r,s}.$$

Observe that (2.99), (2.100) give in particular

$$R^{-1}\left(\tau R\left(\overline{U_1} \cup \overline{U_2}\right) + z\right) \subset RECT,$$

that is, similarly to the simplified setting (see (2.93)), the cylindrical projection R^{-1} maps $\tau R\left(\overline{U_1} \cup \overline{U_2}\right) + z$ into the region in P in which $v_2 = (1, 0)$. Moreover, (2.98) and (2.101) immediately give

$$\begin{aligned} F_1^* &> 0.01B \quad \text{in } RECT, \\ F_1^* &> 8B \qquad \text{in } SBOX. \end{aligned} \tag{2.102}$$

Furthermore, it can be shown (using the properties of the choice of $\varepsilon, d, r, a, s, v_2$ and the decay of H) that

$$v_2 \cdot F^* \geq -1.1\varepsilon B \quad \text{in } \operatorname{supp} v_2. \tag{2.103}$$

Finally, we will make a particular choice of f_2, ϕ_2 and $T > 0$ such that (v_2, f_2, ϕ_2) is a structure on U_2 and the properties (2.50), (2.51) hold. The proof of (2.50) will be in essence similar to the calculation (2.95), but with the inequality (2.94) replaced by (2.103) and a property of the choice of T. The proof of (2.51) is, in a sense, a more elaborate version of the calculation (2.97). Namely, rather than taking any $x \in R\left(\overline{U_1} \cup \overline{U_2}\right)$ we will consider two cases, which correspond to different means of substituting the use of the inequality (2.96):

Case 1. $x \in R\left(\overline{U^{a,r}}\right)$. Then $y \in RECT$ by (2.100) and we will replace (2.96) by the first inequality in (2.102) and the properties of f_2 and T.

Case 2. $x \in R\left(\overline{U} \cup \overline{U^{a',r'}} \cup \overline{U^{a'',r''}} \cup \overline{U_2}\right) \subset R(BOX)$. Then $y \in SBOX$ by (2.99) and we will replace (2.96) by the second inequality in (2.102) and the properties of f_2 and T.

We now present the rigorous version of this explanation.

2.4.2 The copies of U and its structure

Let us consider disjoint "copies" of U and its structure (v, f, ϕ) and arranging these copies into a favourable composition. Namely, for $\alpha \in \mathbb{R}$, $\rho > 0$, $\sigma > 0$ let

$$\begin{aligned} U^{\alpha,\rho} &:= \left\{ (x_1, x_2) \in \mathbb{R}^2 \colon \left(\frac{x_1 - \alpha}{\rho}, \frac{x_2}{\rho} \right) \in U \right\}, \\ v^{\alpha,\rho,\sigma}(x_1, x_2) &:= \sigma\, v\left(\frac{x_1 - \alpha}{\rho}, \frac{x_2}{\rho} \right), \\ f^{\alpha,\rho,\sigma}(x_1, x_2) &:= \sigma\, f\left(\frac{x_1 - \alpha}{\rho}, \frac{x_2}{\rho} \right), \\ \phi^{\alpha,\rho}(x_1, x_2) &:= \phi\left(\frac{x_1 - \alpha}{\rho}, \frac{x_2}{\rho} \right), \\ F^{\alpha,\rho,\sigma}(x_1, x_2) &:= \frac{\sigma^2}{\rho} F\left(\frac{x_1 - \alpha}{\rho}, \frac{x_2}{\rho} \right). \end{aligned} \tag{2.104}$$

(Recall $F = F[v, f]$ is the pressure interaction function.)

Here $\alpha \in \mathbb{R}$ denotes the translation in x_1 direction of U and its structure and ρ denotes the scaling of the variables, see Fig. 2.10. Also, σ denotes the scaling in magnitude of v and f. A direct consequence of the definitions above is that $U^{\alpha,\rho} \Subset P$, $(v^{\alpha,\rho,\sigma}, f^{\alpha,\rho,\sigma}, \phi^{\alpha,\rho})$ is a structure on $U^{\alpha,\rho}$ and $F^{\alpha,\rho,\sigma}$ is a pressure interaction function corresponding to $U^{\alpha,\sigma}$, namely

$$F^{\alpha,\rho,\sigma} = F[v^{\alpha,\rho,\sigma}, f^{\alpha,\rho,\sigma}],$$

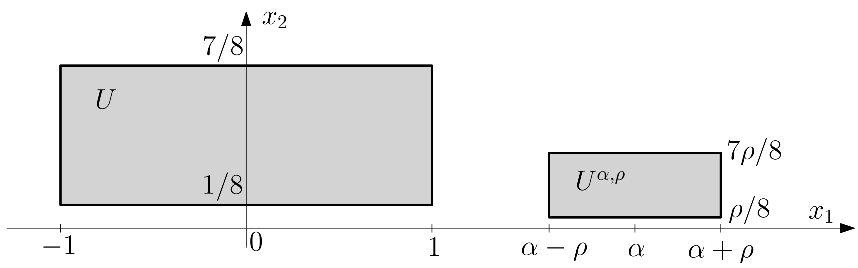

Figure 2.10: The set $U^{\alpha,\rho}$, where $\rho < 1$.

for each choice of $\alpha \in \mathbb{R}$, $\rho, \sigma > 0$. Now let $a', a'' \in \mathbb{R}$, $r', r'', s', s'' > 0$ be such that the sets U, $U^{a',r'}$, $U^{a'',r''}$ have disjoint closures and the function

$$H := F + F^{a',r',s'} + F^{a'',r'',s''} \tag{2.105}$$

satisfies

(i) $H_1(A,0) = 7B$,

(ii) $H_1(x_1,0) \geq -1.005B$,

(iii) $|H(x)| \leq 2C/|x|^4$ for $|x| > 2|A|$.

Such a choice is possible due to the following simple geometric argument (which is sketched in Fig. 2.11). Let s', r' satisfy $(s')^2/r' = 2$ (so that $\max F_1^{a',r',s'}(\cdot,0) = 2B = -\min F_1^{a',r',s'}(\cdot,0)$) and take $r' > 0$ so small that $|F_1^{0,r',s'}(x_1,0)| < 0.001B$ for x_1 such that $F_1(A + x_1, 0) < 0.999B$. Then choose a' such that the maxima of both $F_1(x_1,0)$ and $F_1^{a',r',s'}(x_1,0)$ coincide (at $x_1 = A$). Then, similarly, choose s'', r'' so that $(s'')^2/r'' = 4$ and $r'' > 0$ is small enough so that $|F_1^{0,r'',s''}(x_1,0)| < 0.001B$ for x_1 such that $F_1^{a',r',s'}(A + x_1, 0) < 0.999 \cdot (2B)$, and choose a'' so that the maximum of $F_1^{a'',r'',s''}(x_1,0)$ occurs at $x_1 = A$. This way we obtain (i) and (ii) by construction, while (iii) follows given r' and r'' were chosen large enough. Furthermore, taking r' and r'' small ensures that the sets U, $U^{a',r'}$, $U^{a'',r''}$ have disjoint closures ($r' < 1/8$ and $r'' < r'/8$ suffices, cf. Fig. 2.9).

Thus by specifying $a', a'', r', r'', s', s''$ we added to U two disjoint copies of it such that the total pressure interaction function H has a specific behaviour on the x_1 axis. We now want to specify the behaviour of H on a strip in P near the x_1 axis. That is, by continuity, we see that there exists $E > 0$ (sufficiently small) such that

(iv) the strip $\{0 < x_2 < E\} \subset P$ is disjoint from U, $U^{a',r'}$, $U^{a'',r''}$,

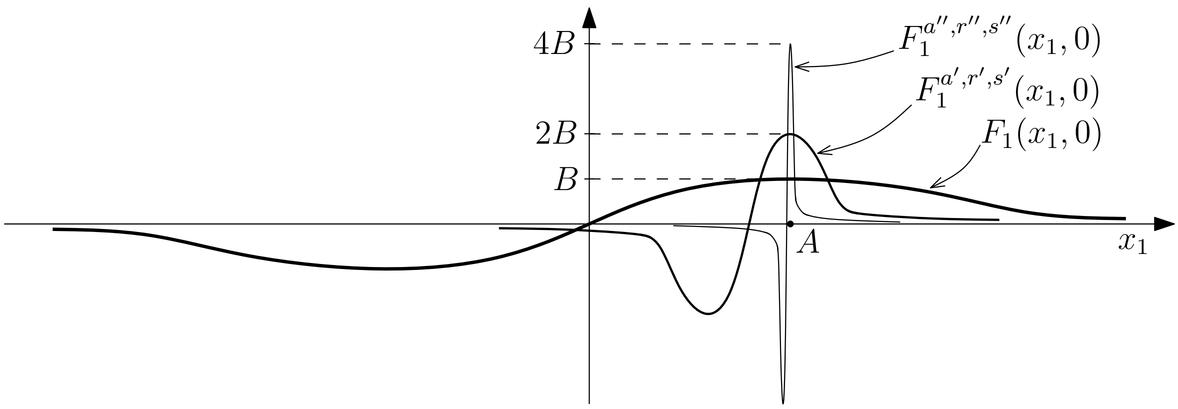

Figure 2.11: The choice of $a', a'', r', r'', s', s''$.

(v) $H(x) \geq -1.01B$ in the strip $\{0 < x_2 < E\}$,

(vi) $H(x) \geq 6.99B$ for $x \in P$ such that $|x_1 - A| < \kappa E$, $0 < x_2 < E$.

Here claim (v) also uses the decay property (iii) of H.

2.4.3 Construction of v_2 and U_2

Now let $\varepsilon > 0$ be a small parameter (whose value we fix below) and let $d, r > 0$ be defined by

$$r \coloneqq E/\varepsilon, \quad d \coloneqq \kappa r.$$

Note that by taking ε small, both r and d become large, and since

$$\kappa = 10^4 C/D > 10^4 \quad \text{we have} \quad d > 10^4 r. \tag{2.106}$$

In fact, ε is the main parameter of the construction and in what follows we will use certain algebraic inequalities, all of which rely on ε being sufficiently small. We gather all these properties here in order to demonstrate that the argument is not circular. Namely, let $\varepsilon > 0$ be sufficiently small that

$$\begin{aligned} &\varepsilon < 1/10, \quad d - r > 2(|A| + \kappa E), \quad r > 10, \quad r > 20|A|, \\ &d > 2\,\mathrm{diam}\left(U \cup U^{a',r'} \cup U^{a'',r''}\right), \quad \varepsilon < \kappa/N, \quad \varepsilon^2 < \frac{BE^4}{2 \cdot 10^6 C}. \end{aligned} \tag{2.107}$$

We now construct v_2 by sharpening the observation from Fig. 2.4. Namely we let v_2 be given by the following lemma.

Lemma 2.14. *Given $d, r, \varepsilon > 0$ such that $d > r$, $\varepsilon < 1/10$ there exists $v_2 = (v_{21}, v_{22}) \in C_0^\infty(P; \mathbb{R}^2)$ such that*

(i) $\operatorname{div}(x_2\, v_2(x_1, x_2)) = 0$,

(ii) $\operatorname{supp} v_2 \subset (-d, d) \times (0.005\varepsilon r, r) \setminus [-(d-r), d-r] \times [\varepsilon r, r/10]$,

(iii) $|v_{22}| < \varepsilon/2$, $-\varepsilon^2 \le v_{21} \le 1$ *with*

$$v_{21} \ge 0, v_{22} = 0 \quad \text{in} \quad [-(d-r), d-r] \times (0, \varepsilon r),$$

(iv) $v_2 = (1, 0)$ *in* $[-(d-r), d-r] \times [0.02\varepsilon r, 0.98\varepsilon r]$.

Before proving the lemma, we note that the construction of such a vector field v_2 is one of the central ideas of the proof of Theorem 2.1. We will shortly see that it is thanks to v_2 that we can overcome the difficulty posed by Warning 2.13. Indeed, we can already see (in part (iv) above) that v_2 keeps constant direction and magnitude in a rectangular-shaped subset of P which is located near the Ox_1 axis, and that $v_2 = O(\varepsilon)$ whenever its direction is different (which we will see in the proof below).

Proof. Let $w \colon P \to \mathbb{R}^2$ be defined by

$$w(x_1, x_2) = \begin{cases} (x_2, 0) & \text{in } R_1, \\ \frac{\varepsilon}{2}(d - x_1, x_2) & \text{in } R_2, \\ -\varepsilon^2 (x_2, 0) & \text{in } R_3, \\ \frac{\varepsilon}{2}(x_1 + d, -x_2) & \text{in } R_4, \\ 0 & \text{in } P \setminus (R_1 \cup R_2 \cup R_3 \cup R_4), \end{cases}$$

where regions R_1, R_2, R_3, R_4 are as indicated in Fig. 2.12. Observe that these regions, and the form of w inside each of them, is defined in way that w is diveregence-free inside each region and $w \cdot n$ is continuous across the boundary between any pair of neighbouring regions, where n denotes the unit normal vector of the boundary. Recall (from a recipe for a structure, Section 2.2.5) that this is sufficient for w to be weakly divergence-free on $\mathbb{R}^2$. Therefore (similarly to the recipe for a structure, see Section 2.2.5) Jw is divergence free, smooth and compactly supported vector field on P, where J denotes any mollification operator. Thus letting

$$v_2 = Jw/x_2$$

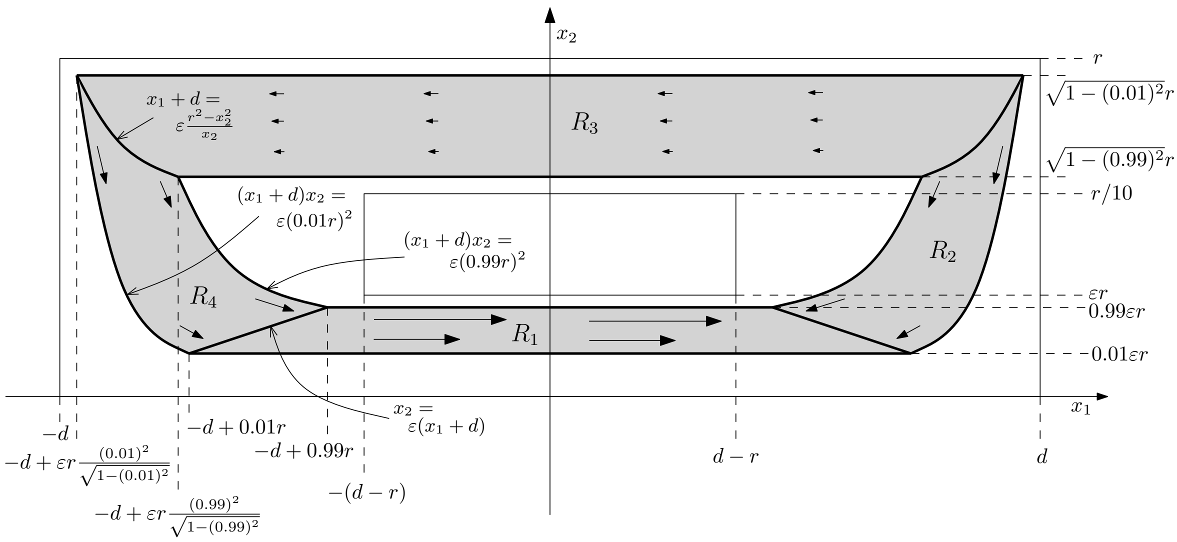

Figure 2.12: The construction of w. The configuration of the curves in $\{x_1 > 0\}$ is defined symmetrically with respect to the x_2 axis. The arrows (inside the grey region) indicate the direction and magnitude of w. Note that some proportions are not conserved on this sketch.

we see that, for sufficiently fine mollification J, v_2 satisfies all the required properties. In particular $v_2 = (1,0)$ in $[-(d-r), d-r] \times [0.02\varepsilon r, 0.98\varepsilon r]$ since affine functions are invariant under mollifications. $\square$

Now let

$$\tau := 0.48\varepsilon, \qquad z := (A, \varepsilon r/2, 0). \tag{2.108}$$

We see that

$$\tau d = \tau \kappa E/\varepsilon < \kappa E. \tag{2.109}$$

Let

$$\begin{aligned} U_2 &:= (-d,d) \times (0.005\varepsilon r, r) \setminus [-(d-r), d-r] \times [\varepsilon r, r/10], \\ BOX &:= [-d,d] \times [0,r], \\ SBOX &:= [A-\kappa E, A+\kappa E] \times [0.02\varepsilon r, 0.98\varepsilon r], \\ RECT &:= [-(d-r), d-r] \times [0.02\varepsilon r, 0.98\varepsilon r], \end{aligned} \tag{2.110}$$

see Fig. 2.13.

Note that $\operatorname{supp} v_2 \subset U_2$ by construction (see Lemma 2.14 (ii)) and that $SBOX \subset RECT$ by the second inequality in (2.107). Moreover,

$$R^{-1}(\tau R(BOX) + z) \subset SBOX. \tag{2.111}$$

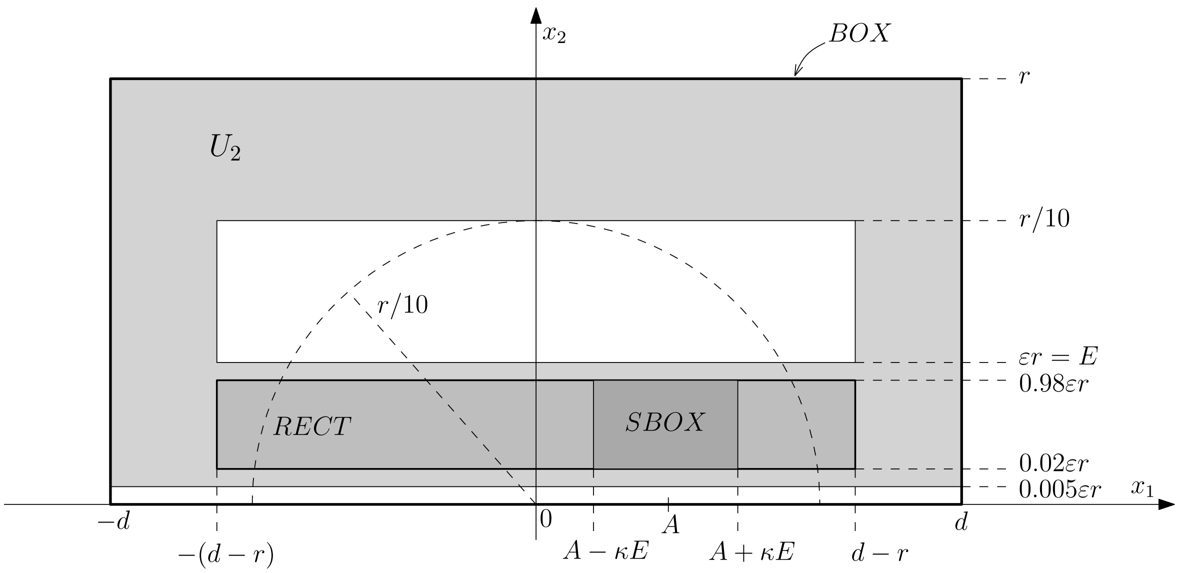

Figure 2.13: The sets U_2, BOX, $RECT$ and $SBOX$. Note that proportions are not conserved on this sketch.

Indeed, since $\tau r < \varepsilon r/2$ we observe that the set on the left-hand side is simply

$$[A-\tau d, A+\tau d] \times [\varepsilon r/2 - \tau r, \varepsilon r/2 + \tau r] = [A-\tau d, A+\tau d] \times [0.02\varepsilon r, 0.98\varepsilon r] \subset SBOX,$$

where the inclusion follows from (2.109). What is more, the sets U, $U^{a',r'}$, $U^{a'',r''}$ are "encompassed" by U_2, that is

$$U \cup U^{a',r'} \cup U^{a'',r''} \subset (-(d-r), d-r) \times (\varepsilon r, r/10), \tag{2.112}$$

see Fig. 2.14. This property is clear from the identity $\varepsilon r = E$ and property (iv) of the choice of E (so that the strip $\{0 < x_2 < \varepsilon r\}$ is "below" these sets), the third inequality in (2.107) (so that the half-plane $\{x_2 > r/10\}$ is above U), and the fifth inequality in (2.107), which gives

$$d - r > \operatorname{diam}\left(U \cup U^{a',r'} \cup U^{a'',r''}\right)$$

(so that the length in the x_1 direction of the set $U \cup U^{a',r'} \cup U^{a'',r''}$ is less than $d-r$; recall also $d > 2r$ by (2.106)). Furthermore, properties (v) and (vi) of H (and the trivial inequality $0.98\varepsilon r \le E$) immediately give that

$$\begin{cases} H_1(x) \ge -1.01B & \text{in } (-(d-r), d-r) \times (0, \varepsilon r) \supset RECT, \\ H_1(x) \ge 6.99B & \text{in } SBOX. \end{cases} \tag{2.113}$$

2.4.4 Construction of U_1 and its structure

We will add one more copy of U (and its structure) to the collection U, $U^{a',r'}$, $U^{a'',r''}$ (and the corresponding collection of structures). Namely let

$$a := -\kappa r/\varepsilon, \quad \frac{s^2}{r} := 1.04\left(-\frac{a}{r}\right)^4 B/D, \tag{2.114}$$

and consider $U^{a,r}$ with structure $(v^{a,r,s}, f^{a,r,s}, \phi^{a,r})$. In this way, the pressure interaction function

$$F^{a,r,s} = F[v^{a,r,s}, f^{a,r,s}]$$

is of particular size in the whole of BOX, which we make precise in the following lemma.

Lemma 2.15.

$$1.03B \le F_1^{a,r,s} \le 1.05B \quad \textit{and} \quad |F_2^{a,r,s}| \le 0.01\varepsilon B \quad \textit{in } \mathrm{BOX}.$$

Proof. As for $F_1^{a,r,s}$ let $n := -a/r$ and observe that the sixth inequality in (2.107) gives $n \ge N$. Thus, since $|x_2|/r \le 1$ and

$$\left|\frac{x_1 - a}{r} - n\right| = \frac{|x_1|}{r} \le \frac{d}{r} = \kappa$$

Lemma 2.9 (v) gives

$$\left|F_1\left(\frac{x_1 - a}{r}, \frac{x_2}{r}\right) - n^{-4}D\right| \le 0.001 n^{-4} D.$$

(Recall (from the paragraph preceeding Section 2.4.1) that $F = (F_1, F_2)$ denotes the pressure interaction function corresponding to U and structure (v, f, ϕ), that is $F = F[v, f]$.) Therefore, since

$$F_1\left(\frac{x_1 - a}{r}, \frac{x_2}{r}\right) = \frac{r}{s^2} F_1^{a,r,s}(x_1, x_2) = \frac{n^{-4}D}{1.04B} F_1^{a,r,s}(x_1, x_2)$$

(recall (2.104) and (2.114)), we can multiply the last inequality by $1.04B/(n^{-4}D)$ to obtain

$$|F_1^{a,r,s}(x) - 1.04B| \le 0.001(1.04B) < 0.01B \quad \text{for } x \in BOX.$$

As for $F_2^{a,r,s}$ let $(x_1, x_2) \in BOX$ and use Lemma 2.9 (iv), the Mean Value Theorem and Lemma 2.9 (iii) to write

$$\begin{aligned} \frac{r}{s^2}\left|F_2^{a,r,s}(x_1, x_2)\right| &= \left|F_2\left(\frac{x_1 - a}{r}, \frac{x_2}{r}\right) - F_2\left(\frac{x_1 - a}{r}, 0\right)\right| \\ &\le \left|\nabla F_2\left(\frac{x_1 - a}{r}, \xi\right)\right| \left|\frac{x_2}{r}\right| \le C\left|\frac{x_1 - a}{r}\right|^{-5}, \end{aligned}$$

where $\xi \in (0,1)$. Thus, since the triangle inequality and the fact $\varepsilon < 1/2$ give

$$\frac{|x_1 - a|}{r} \geq \frac{|a|}{r} - \frac{|x_1|}{r} \geq \frac{|a|}{r} - \frac{d}{r} = \kappa \left(\frac{1}{\varepsilon} - 1 \right) \geq \frac{\kappa}{2\varepsilon},$$

we obtain (recalling (2.114) and that $\kappa = 10^4 C/D$, see (2.106))

$$|F_2^{a,r,s}(x_1,x_2)| \leq \left(2^5 \frac{1.04C}{D\kappa} \right) \varepsilon B = \frac{32 \cdot 1.04}{10^4} \varepsilon B < 0.01 \varepsilon B. \qquad \square$$

Thus letting

$$\begin{aligned} U_1 &:= U \cup U^{a',r'} \cup U^{a'',r''} \cup U^{a,r}, \\ f_1 &:= f + f^{a',r',s'} + f^{a'',r'',s''} + f^{a,r,s}, \\ v_1 &:= v + v^{a',r',s'} + v^{a'',r'',s''} + v^{a,r,s}, \\ \phi_1 &:= \phi + \phi^{a',r'} + \phi^{a'',r''} + \phi^{a,r} \end{aligned}$$

we obtain a structure (v_1, f_1, ϕ_1) on U_1, and denoting by F^* the total pressure interaction function,

$$F^* := F[v_1, f_1] = F + F^{a',r',s'} + F^{a'',r'',s''} + F^{a,r,s} = H + F^{a,r,s},$$

we see that the above lemma and (2.113) give

$$\begin{cases} F_1^* \geq 0.01B & \text{in } (-(d-r), d-r) \times (0, \varepsilon r) \supset RECT, \\ F_1^* \geq 8B & \text{in } SBOX. \end{cases} \tag{2.115}$$

Moreover, the properties of H (the "joint" pressure interaction function of U, $U^{a',r'}$ and $U^{a'',r''}$, recall (2.105)), v_2, the smallness of ε (recall (2.107)) and the lemma above give

$$v_2 \cdot F^* \geq -1.1 \varepsilon B \qquad \text{in } BOX, \tag{2.116}$$

which we now verify. The claim for $x \in BOX \setminus \operatorname{supp} v_2$ follows trivially. For $x \in \operatorname{supp} v_2$ consider two cases.

Case 1. $|x| < r/10$. In this case observe that since $d > 10^4 r$ (see (2.106)) we have $d - r > r/10$ and so $x \in (-(d-r), d-r) \times (0, \varepsilon r)$ (cf. Fig. 2.13). Thus, $v_{21}(x) \geq 0$, $v_{22}(x) = 0$ by construction of v_2 (see Lemma 2.14 (iii)), and so (2.115) gives

$$v_2(x) \cdot F^*(x) = v_{21}(x) F_1^*(x) \geq 0.01 B v_{21}(x) > 0 > -1.1 \varepsilon B.$$

Case 2. $|x| \geq r/10$. Since $r > 20|A|$ (see (2.107)), in this case $|x| \geq 2|A|$, and so property (iii) of H and the last property in (2.107) give

$$|H(x)| \leq 2C/|x|^4 \leq 2C \left(\frac{10}{r} \right)^4 = 2 \cdot 10^4 C \varepsilon^4 / E^4 < 0.01 \varepsilon^2 B. \tag{2.117}$$

This, the properties $-\varepsilon^2 \le v_{21} \le 1$, $|v_{22}| < \varepsilon/2$ (see Lemma 2.14 (iii)) and Lemma 2.15 give

$$\begin{aligned} v_2(x) \cdot F^*(x) &= v_2(x) \cdot H(x) + v_{21}(x) F_1^{a,r,s}(x) + v_{22}(x) F_2^{a,r,s}(x) \\ &\ge -2(0.01\varepsilon^2 B) - \varepsilon^2(1.05B) - \frac{\varepsilon}{2}(0.01B\varepsilon) \\ &= -\varepsilon^2 B(0.02 + 1.05 + 0.005) \ge -1.1\varepsilon^2 B. \end{aligned}$$

Thus we obtain (2.116), as required.

Moreover, since $U^{a,r} = (a-r, a+r) \times (r/8, 7r/8)$, we see that

$$U^{a,r} \text{ is located "to the left" of } BOX, \tag{2.118}$$

that is $a + r < -d$ (see Fig. 2.14), which can be verified as follows. Since $\varepsilon < 1/10$ (recall (2.107)) and $\kappa > 1$ (recall (2.106)) we trivially obtain

$$\kappa \left(\frac{1}{\varepsilon} - 1 \right) > 1,$$

which, multiplied by r, gives

$$-\frac{\kappa r}{\varepsilon} + r < -\kappa r,$$

that is $a + r < -d$, as required. Thus, taking into account (2.112) we see that $\overline{U_1}$ and $\overline{U_2}$ are disjoint (see Fig. 2.14), which is one of the requirements of the geometric arrangement.

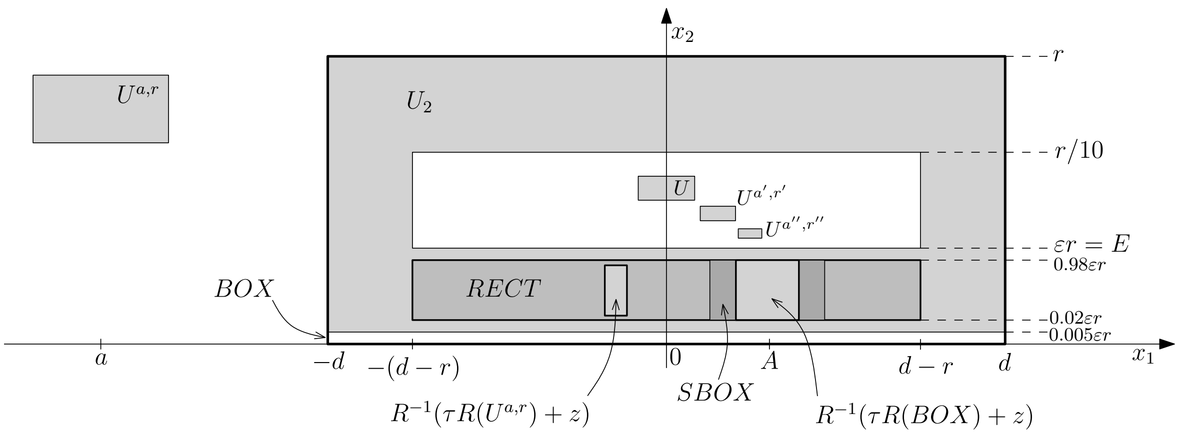

Figure 2.14: The geometric arrangement (cf. Fig. 2.5).

Furthermore note that

$$R^{-1}\left(\tau R\left(\overline{U^{a,r}}\right) + z\right) \subset RECT, \tag{2.119}$$

see Fig. 2.14. Indeed, since $U^{a,r} = (a-r, a+r) \times (r/8, 7r/8)$ (recall (2.104)) we see that the set on the left-hand side is simply

$$[\tau(a-r) + A, \tau(a+r) + A] \times [\varepsilon r/2 - 7\tau r/8, \varepsilon r/2 + 7\tau r/8]$$

The second of these intervals is contained in $[\varepsilon r/2 - \tau r, \varepsilon r/2 + \tau r] = [0.02\varepsilon r, 0.98\varepsilon r]$, where we recalled that $\tau = 0.48\varepsilon$ (see (2.108)). Thus (2.119) follows if the first of the intervals is contained in $[-(d-r), d-r]$, that is if

$$|\tau a + A| \leq d - r - \tau r.$$

This last inequality follows from the fourth inequality in (2.107) and the facts that $\kappa > 10^4$ (recall (2.106)) and $\tau < 1$, by writing

$$\begin{aligned} |\tau a + A| &\leq \tau |a| + |A| = 0.48\kappa r + |A| \\ &\leq 0.48\kappa r + 0.05 r < \kappa r/2 < (\kappa - 2) r = d - 2r < d - r - \tau r. \end{aligned}$$

The inclusions (2.111) and (2.119) combine to give

$$R^{-1}(\Gamma(G)) \subset RECT \subset \overline{U_2},$$

and thus

$$\Gamma(G) \subset R(\overline{U_2}) \subset G$$

(recall $G = R(\overline{U_1} \cup \overline{U_2})$), as required by the geometric arrangement, which is sketched in Fig. 2.14.

2.4.5 Construction of f_2, ϕ_2, T and conclusion of the arrangement

It remains to construct f_2, ϕ_2, T such that (v_2, f_2, ϕ_2) is a structure on U_2 and properties (2.50), (2.51) hold, that is

$$f_2^2 + T v_2 \cdot F^* > |v_2|^2 \qquad \text{in } U_2,$$

and

$$f_2^2(y) + T v_2(y) \cdot F^*(y) > \tau^{-2} \left(f_1(R^{-1}x) + f_2(R^{-1}x) \right)^2$$

for

$$y = R^{-1}(\tau x + z), \quad x \in R\left(\overline{U_1} \cup \overline{U_2}\right),$$

respecitvely.

To this end, note that since U_2 is a rectangular ring, we can (as in Section 2.2.5) use Theorem 2.8 to obtain $f_2 \in C_0^\infty(P;[0,1])$ such that $\operatorname{supp} f_2 = \overline{U_2}$, $f_2 > 0$ in U_2, $f_2 = \mu$ on $\operatorname{supp} v_2$, $Lf_2 > 0$ at points of U_2 of sufficiently small distance to ∂U_2, where $\mu > 100$ is sufficiently large such that

$$\mu \geq 100 \| f_1 \|_\infty . \tag{2.120}$$

Following the recipe for a structure (Section 2.2.5) we let $\phi_2 \in C_0^\infty(U_2;[0,1])$ be a cut off function such that $\phi_2 = 1$ in $\operatorname{supp} v_2$ and $Lf_2 > 0$ in $U_2 \setminus \{\phi_2 = 1\}$. Thus (v_2, f_2, ϕ_2) is a structure on U_2. We now let

$$T \coloneqq \frac{\mu^2 - 5}{1.1\varepsilon^2 B} \geq \frac{0.9\mu^2}{1.1\varepsilon^2 B} \tag{2.121}$$

and we verify (2.50) and (2.51).

Using (2.116) and the fact that $|v_2| \leq 2$ (recall Lemma 2.14 (iii)) we immediately obtain (2.50) by writing

$$f_2^2 + T v_2 \cdot F^* \geq \mu^2 - 1.1\varepsilon^2 B T = 5 > |v_2|^2 \qquad \text{in } \operatorname{supp} v_2,$$

and the claim in $U_2 \setminus \operatorname{supp} v_2$ follows trivially from positivity of f_2 in U_2.

As for (2.51), we need to show

$$f_2^2(y) + T v_2(y) \cdot F^*(y) > \tau^{-2} \left(f_1(R^{-1}x) + f_2(R^{-1}x) \right)^2$$

for

$$y = R^{-1}(\tau x + z), \quad x \in R\left(\overline{U_1} \cup \overline{U_2}\right).$$

To this end fix $x \in R\left(\overline{U_1} \cup \overline{U_2}\right)$. Since

$$\overline{U_1} \cup \overline{U_2} = \left(\overline{U_2} \cup \overline{U} \cup \overline{U^{a',r'}} \cup \overline{U^{a'',r''}} \right) \cup \overline{U^{a,r}}$$

we consider two cases.

Case 1. $x \in R\left(\overline{U_2} \cup \overline{U} \cup \overline{U^{a',r'}} \cup \overline{U^{a'',r''}} \right)$.

Then $R^{-1}x \in BOX$ and hence $y \in SBOX$ by (2.111). Thus $v_2(y) = (1,0)$ and $F_1^*(y) \geq 8B$ in $SBOX$ (see Lemma 2.14 (iv) and (2.115)) and, using (2.121) and (2.120),

$$\begin{aligned} f_2^2(y) + T v_2(y) \cdot F^*(y) &\geq 8TB \geq \frac{7.2}{1.1} \left(\frac{\mu}{\varepsilon}\right)^2 > \left(\frac{1.01}{0.48}\right)^2 \left(\frac{\mu}{\varepsilon}\right)^2 \\ &= \tau^{-2} (1.01\mu)^2 \geq \tau^{-2} \left(\|f_2\|_\infty + \|f_1\|_\infty \right)^2 . \end{aligned}$$

Case 2. $x \in R\left(\overline{U^{a,r}}\right)$. Then $f_2(R^{-1}x) = 0$ (since $R^{-1}x \in \overline{U^{a,r}} \subset \overline{U_1}$ and $\overline{U_1}$, $\overline{U_2}$ are disjoint) and $y \in RECT$ (see (2.119)). Therefore, $v_2(y) = (1,0)$, $F_1^*(y) \geq 0.02B$ (by Lemma 2.14 (iv) and (2.115)) and so, using (2.121) and (2.120),

$$f_2^2(y) + Tv_2(y) \cdot F^*(y) \geq 0.01TB \geq \frac{0.009}{1.1}\left(\frac{\mu}{\varepsilon}\right)^2 > \left(\frac{0.01}{0.48}\right)^2 \left(\frac{\mu}{\varepsilon}\right)^2$$
$$= \tau^{-2}(0.01\mu)^2 \geq \tau^{-2}\|f_1\|_\infty^2 \geq \tau^{-2}\left(f_1(R^{-1}x) + f_2(R^{-1}x)\right)^2.$$

Hence we obtain (2.51). This concludes the construction of geometric arrangement, and so also the proof of Theorem 2.1.

2.5 Appendix

2.5.1 The function f supported in $\overline{U}$ and with $Lf > 0$ near ∂U

Here we show that for any set $U \Subset P$ of the shape of a rectangle or a "rectangle ring", that is $U = V \setminus \overline{W}$ for some open rectangles V, W with $W \Subset V$, and any $\eta > 0$ there exists $\delta \in (0,\eta)$ and $f \in C_0^\infty(P;[0,1])$ such that

$$\operatorname{supp} f = \overline{U}, \quad f > 0 \text{ in } U \text{ with } f = 1 \text{ on } U_\eta$$

and

$$Lf > 0 \quad \text{ in } U \setminus U_\delta.$$

The claim follows from Lemma 2.18 below (which corresponds to the case of a rectangle) and from Lemma 2.19 (which corresponds to the case of a rectangle ring).

We will need a certain generalisation of the Mean Value Theorem. For $f\colon \mathbb{R} \to \mathbb{R}$ let $f[a,b]$ denote the finite difference of f on $[a,b]$,

$$f[a,b] \coloneqq \frac{f(a)-f(b)}{a-b}$$

and let $f[a,b,c]$ denote the finite difference of $f[\cdot,b]$ on $[a,c]$,

$$f[a,b,c] \coloneqq \left(\frac{f(a)-f(b)}{a-b} - \frac{f(c)-f(b)}{c-b}\right)/(a-c).$$

Lemma 2.16 (generalised Mean Value Theorem)**.** *If $a < b < c$, f is continuous in $[a,c]$ and twice differentiable in (a,c) then there exists $\xi \in (a,c)$ such that $f[a,b,c] = f''(\xi)/2$.*

Proof. We follow the argument of Theorem 4.2 in Conte & de Boor (1972). Let

$$p(x) \coloneqq f[a,b,c](x-b)(x-c) + f[b,c](x-c) + f(c).$$

Then p is a quadratic polynomial approximating f at a,b,c, that is $p(a) = f(a)$, $p(b) = f(b)$, $p(c) = f(c)$. Thus the error function $e(x) \coloneqq f(x) - p(x)$ has at least 3 zeros in $[a,c]$. A repeated application of Rolle's theorem gives that e'' has at least one zero in (a,c). In other words, there exists $\xi \in (a,c)$ such that $f''(\xi) = p''(\xi) = 2f[a,b,c]$. $\square$

Corollary 2.17. *If $f \in C^3$ is such that $f = 0$ on $(a-\delta, a]$ and $f''' > 0$ on $(a, a+\delta)$ for some $a \in \mathbb{R}$, $\delta > 0$ then*

$$\begin{cases} f''(x) > 0, \\ 0 < f'(x) < (x-a) f''(x), \\ f(x) < (x-a)^2 f''(x) \end{cases} \qquad \text{for } x \in (a, a+\delta).$$

Similarly, if $g = 0$ on $[a, a+\delta)$ and $g''' < 0$ on $(a-\delta, a)$ then

$$\begin{cases} g''(x) > 0, \\ 0 > g'(x) > (x-a) g''(x), \\ g(x) < (x-a)^2 g''(x) \end{cases} \qquad \text{for } x \in (a-\delta, a).$$

Proof. Since $f''' > 0$ on $(a, a+\delta)$ we see that f'' is positive and increasing on this interval and so the first two claims follow for f from the Mean Value Theorem. The last claim follows from the lemma above by noting that $2a - x \in (a-\delta, a]$,

$$\begin{aligned} f(x) &= f(2a-x) - 2f(a) + f(x) = 2(x-a)^2 f[2a-x, a, x] \\ &= (x-a)^2 f''(\xi) < (x-a)^2 f''(x), \end{aligned}$$

where $\xi \in (2a-x, x)$. The claim for g follows by considering $f(x) \coloneqq g(2a-x)$. $\square$

We now show the claim in the case of U in the shape of a rectangle.

Lemma 2.18 (The cut-off function on a rectangle)**.** *Let $U \Subset P$ be an open rectangle, that is $U = (a_1, b_1) \times (a_2, b_2)$ for some $a_1, a_2, b_1, b_2 \in \mathbb{R}$ with $b_1 > a_1$, $b_2 > a_2 > 0$. Given $\eta > 0$ there exists $\delta \in (0, \eta)$ and $f \in C_0^\infty(P; [0,1])$ such that*

$$\operatorname{supp} f = \overline{U}, \quad f > 0 \text{ in } U \text{ with } f = 1 \text{ on } U_\eta,$$

$$Lf > 0 \quad \text{in } U \setminus U_\delta,$$

and f is symmetric with respect to the vertical axis of U, that is

$$f\left(\frac{a_1+b_1}{2}-x_1,x_2\right)=f\left(\frac{a_1+b_1}{2}+x_1,x_2\right),\quad (x_1,x_2)\in P.$$

Proof. By assumption

Let $f_1,f_2\in C_0^\infty(\mathbb{R};[0,1])$ be such that $\operatorname{supp} f_i=[a_i,b_i]$, $f_i>0$ on (a_i,b_i) with $f_i=1$ on $[a_i+\eta,b_i-\eta]$,

$$f_i'''>0 \text{ on } (a_i,a_i+\varepsilon)\quad\text{and}\quad f_i'''<0 \text{ on } (b_i-\varepsilon,b_i),\quad i=1,2,$$

for some $\varepsilon\in(0,\eta)$. (Take for instance f_i's such that

$$f_i(x)=\begin{cases}0 & x\le a_i,\\ \exp\left(-(x-a_i)^{-2}\right) & x\in(a_i,a_i+\varepsilon),\\ 1 & x\in(a_i+\eta,b_i-\eta),\\ \exp\left(-(b_i-x)^{-2}\right) & x\in(b_i-\varepsilon,b_i),\\ 0 & x\ge b_i,\end{cases}$$

where $\varepsilon\in(0,\eta)$ is sufficiently small such that $f_i\le 1$ on each of the intervals above, and define f_i on the remaining intervals $[a_i+\varepsilon,a_i+\eta]$, $[b_i-\eta,b_i-\varepsilon]$ in the way such that $f_i\in C^\infty$, $f_i\le 1$ and

$$f_i\left(\frac{a_i+b_i}{2}-x\right)=f_i\left(\frac{a_i+b_i}{2}+x\right),\quad x\in\mathbb{R},$$

$i=1,2$, see Fig. 2.15)

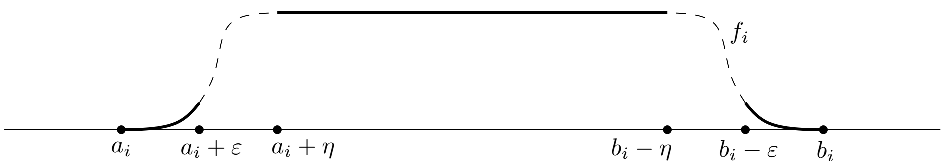

Figure 2.15: The f_i's, $i=1,2$.

Let $f(x_1,x_2)\coloneqq f_1(x_1)f_2(x_2)$. Clearly $\operatorname{supp} f=\overline{U}$, $f>0$ in U, $f=1$ on U_η and the last requirement of the lemma is satisfied due to the equality above. It remains to show that $Lf>0$ on $U\setminus U_\delta$ for some $\delta>0$. Let

$$\begin{aligned}g_1(x_1)&\coloneqq f_1''(x_1),\\ g_2(x_2)&\coloneqq f_2''(x_2)+f_2'(x_2)/x_2-f_2(x_2)/x_2^2.\end{aligned}$$

Then

$$\begin{aligned} Lf(x_1,x_2) &= f_1''(x_1)f_2(x_2) + f_1(x_1)f_2''(x_2) + f_1(x_1)f_2'(x_2)/x_2 - f_1(x_1)f_2(x_2)/x_2^2 \\ &= g_1(x_1)f_2(x_2) + f_1(x_1)g_2(x_2). \end{aligned}$$

Claim: There exists $d>0$ such that

$$g_2 > f_2''/4 > 0 \qquad \text{on } (a_2, a_2+d) \cup (b_2-d, b_2)\,.$$

The claim follows from the corollary of the generalised Mean Value Theorem (see Corollary 2.17) by writing, for $d>0$ small such that $d<a_2/2$, $d<\varepsilon$ and $d/(b_2-d) < 1/2$,

$$\begin{aligned} g_2(x_2) &> f_2''(x_2) - f_2(x_2)/x_2^2 > f_2''(x_2)\left(1-\left(\frac{x_2-a_2}{x_2}\right)^2\right) \\ &> f_2''(x_2)\left(1-\left(\frac{d}{a_2}\right)^2\right) > \frac{3}{4}f_2''(x_2) > \frac{1}{4}f_2''(x_2) > 0 \end{aligned}$$

for $x_2 \in (a_2, a_2+d)$, and

$$\begin{aligned} g_2(x_2) &= f_2''(x_2) + f_2'(x_2)/x_2 - f_2(x_2)/x_2^2 > f_2''(x_2)\left(1+\frac{x_2-b_2}{x_2} - \left(\frac{x_2-b_2}{x_2}\right)^2\right) \\ &> f_2''(x_2)\left(1-\frac{d}{b_2-d} - \left(\frac{d}{b_2-d}\right)^2\right) > f_2''(x_2)/4 > 0 \end{aligned}$$

for $x_2 \in (b_2-d, b_2)$.

Using the claim we see that g_i, f_i are positive on $(a_i, a_i+d)\cup(b_i-d,b_i)$, $i=1,2$. Thus

$$Lf>0 \quad \text{in } ((a_1,a_1+d)\cup(b_1-d,b_1)) \times ((a_2,a_2+d)\cup(b_2-d,b_2))\,,$$

that is in the "d-corners" of U, see Fig. 2.16.

Now let $m, M>0$ be small such that $f_i > m$, $|g_i|<M$ in $[a_i+d, b_i-d]$, $i=1,2$. Let $\delta\in(0,d)$ be such that $m/4-\delta^2 M>0$. The proof of the lemma is complete when we show that

$$Lf>0 \quad \text{in } [a_i+d,b_i-d]\times((a_j,a_j+\delta)\cup(b_j-\delta,b_j))\,,\ (i,j)=(1,2),(2,1),$$

that is in the "δ-strips" at ∂U between the d-corners, see Fig. 2.16. Let $x_1 \in [a_1 + d, b_1-d]$ and $x_2\in(a_2,a_2+\delta)$. Then $g_1(x_1) > -M$, $g_2(x_2) > f_2''(x_2)$ (from *Claim*),

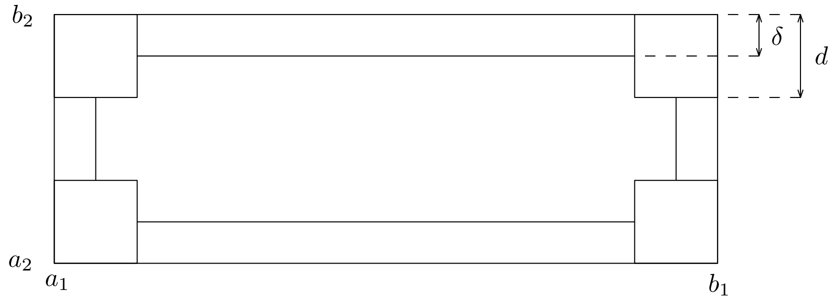

Figure 2.16: The "d-corners" and "δ-stripes".

$f_2(x_2) < (x_2-a_2)^2 f_2''(x_2)$ (from the generalised Mean Value Theorem, see Corollary 2.17), $f_1(x_1) > m$, and so

$$\begin{aligned} Lf(x_1,x_2) &= g_1(x_1)f_2(x_2) + f_1(x_1)g_2(x_2) > -Mf_2(x_2) + f_1(x_1)f_2''(x_2)/4 \\ &> f_2''(x_2)\left(-M(x_2-a_2)^2 + m/4\right) > f_2''(x_2)\left(m/4 - M\delta^2\right) > 0. \end{aligned}$$

As for $x_2 \in (b_2-\delta, b_2)$, simply replace a_2 in the above calculation by b_2. The opposite case, that is the case $x_1 \in (a_1, a_1+\delta) \cup (b_1-\delta, b_1)$, $x_2 \in [a_2+d, b_2-d]$, follows similarly. □

Let

$$U^\delta \coloneqq \{x \in \mathbb{R}^2 \colon \mathrm{dist}\,(x,U) < \delta\}$$

denote the *δ-neighbourhood* of U. We now extend the above lemma to the case of U in the shape of a "rectangular ring".

Lemma 2.19 (The cut-off function on a rectangular ring)**.** *If $U \Subset P$ is a* rectangular ring*, that is $U = V \setminus \overline{W}$ where V, W are open rectangles with $W \Subset V$, then the assertion of the last lemma is valid.*

Proof. It is enough to show that there exist $\delta > 0$ and $f \in C^\infty(P;[0,1])$ such that $f = 0$ on $\overline{W}$, $f > 0$ outside $\overline{W}$ with $f = 1$ outside W^η and

$$Lf > 0 \quad \text{in } W^\delta \setminus \overline{W}.$$

Then the lemma follows by letting

$$g \coloneqq \begin{cases} \widetilde{f} & \text{on } P \setminus W^\eta, \\ f & \text{on } W^\eta, \end{cases}$$

where $\widetilde{f}$ is from the previous lemma applied to V. Denote $W=(a_1,b_1)\times(a_2,b_2)$ for some $a_1,a_2,b_1,b_2\in\mathbb{R}$ with $b_1>a_1$ and $b_2>a_2>0$. Let $f_1,f_2\in C^\infty(\mathbb{R};[0,1])$ be such that $f_i=1$ outside $(a_i-\eta/2,b_i+\eta/2)$, $f_i=0$ on (a_i,b_i) and

$$f_i'''<0 \text{ on } (a_i-\varepsilon,a_i) \quad \text{on} \quad f_i'''>0 \text{ on } (b_i,b_i+\varepsilon), \quad i=1,2,$$

for some $\varepsilon\in(0,\eta/2)$. (Such functions can be constructed by a use of the exponential function, as in the previous lemma, see also Fig. 2.17) Let $f(x_1,x_2):=f_1(x_1)f_2(x_2)$.

Figure 2.17: The f_i's, $i=1,2$ (cf. Fig. 2.15).

Then $f=0$ on $\overline{W}$, $f>0$ outside $\overline{W}$ with $f=1$ outside W^η. It remains to show that $Lf>0$ in $W^\delta\setminus\overline{W}$ for some $\delta>0$. Note that

$$\begin{aligned} Lf(x_1,x_2)&=\left(f_1''(x_1)-f_1(x_1)/x_2^2\right)+\left(f_2''(x_2)+f_2'(x_2)/x_2-f_2(x_2)/x_2^2\right)\\ &=: g_1(x_1,x_2)+g_2(x_2). \end{aligned}$$

As in *Claim* in the proof of the previous lemma we see that

$$g_2>f_2''/4>0 \quad \text{in } (a_2-\delta,a_2)\cup(b_2,b_2+\delta)$$

for sufficiently small $\delta>0$. Thus since f_2 vanishes on $[a_2,b_2]$ we see that

$$g_2\geq 0 \text{ on } (a_2-\delta,b_2+\delta) \quad \text{with} \quad g_2>0 \text{ outside } [a_2,b_2]. \tag{2.122}$$

As for g_1 let δ be such that $\delta/(a_2-\delta)<1/2$. Then, using the corollary of the generalised Mean Value Theorem (Corollary 2.17), we obtain for any $x_2>a_2-\delta$

$$\begin{aligned} g_1(x_1,x_2)&=f_1''(x_1)-f_1(x_1)/x_2^2>f_1''(x_1)\left(1-\left(\frac{x_1-a_1}{x_2}\right)^2\right)\\ &>f_1''(x_1)\left(1-\left(\frac{\delta}{a_2-\delta}\right)^2\right)>\frac{3}{4}f_1''(x_1)>0 \end{aligned}$$

for $x_1\in(a_1-\delta,a_1)$. As for $x_1\in(b_1,b_1+\delta)$ replace a_1 in the above calculation by b_1. Thus, since f_1 vanishes on $[a_1,b_1]$ we see that for each $x_2>a_2-\delta$

$$g_1(\cdot,x_2)\geq 0 \text{ on } (a_1-\delta,b_1+\delta) \quad \text{with} \quad g_1(\cdot,x_2)>0 \text{ outside } [a_1,b_1].$$

This and (2.122) give

$$Lf \geq 0 \text{ on } W^\delta \quad \text{ with } \quad Lf > 0 \text{ outside } \overline{W},$$

as required. □

2.5.2 Preliminary calculations

Let $\phi \in [0, 2\pi)$ and let $R \coloneqq R_\phi$ for brevity of notation. We can represent R in the matrix form

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix}.$$

Note that R is orthogonal, so that $R^T R = I$, where I denotes the identity matrix. Denote

$$\nabla u \coloneqq \begin{pmatrix} \partial_1 u_1 & \partial_2 u_1 & \partial_2 u_1 \\ \partial_1 u_2 & \partial_2 u_2 & \partial_3 u_2 \\ \partial_1 u_3 & \partial_2 u_3 & \partial_3 u_3 \end{pmatrix}$$

If $u(Rx) = Ru(x)$ and $q(Rx) = q(x)$ we write

$$\begin{aligned} ((u\cdot\nabla)u)(Rx) &= \nabla u(Rx)u(Rx) = R\nabla(u(Rx))Ru(x) = R\nabla(Ru(x))Ru(x) \\ &= R\,R^T\nabla u(x)Ru(x) = R\nabla u(x)u(x) = R((u\cdot\nabla)u)(x), \end{aligned}$$

$$\begin{aligned} |u|^2(Rx) &= u(Rx)\cdot u(Rx) = (Ru(x))\cdot(Ru(x)) = u(x)\cdot u(x) = |u|^2(x), \\ \operatorname{div} u(x) &= \operatorname{div}(u(x)) = \operatorname{div}(R^T u(Rx)) = \sum_{i,j} \partial_i \left(R_{ji} u_j(Rx)\right) \\ &= \sum_{i,j,k} R_{ji}\partial_k u_j(Rx) R_{ki} = \sum_{j,k} \delta_{jk}\partial_k u_j(Rx) = \operatorname{div} u(Rx), \end{aligned}$$

where δ_{jk} denotes the Kronecker delta.

$$(u\cdot\nabla q)(Rx) = u(Rx)\cdot\nabla q(Rx) = (Ru(x))\cdot(R\nabla(q(Rx))) = u(x)\cdot\nabla(q(x)) = (u\cdot\nabla)q.$$

By taking $q \coloneqq |u|^2$ we obtain

$$(u\cdot\nabla|u|^2)(Rx) = (u\cdot\nabla|u|^2)(x).$$

Also, for each $k \in \{1,2,3\}$

$$\begin{aligned}
\Delta u_k(x = \Delta(u_k(x)) &= \sum_j \Delta(R_{jk}u_j(Rx)) = \sum_{i,j} \partial_i\partial_i(R_{jk}u_j(Rx)) \\
&= \sum_{i,j,l} R_{jk}R_{li}\partial_i(\partial_l u_j(Rx)) = \sum_{i,j,l,m} R_{jk}R_{li}R_{mi}\partial_m\partial_l u_j(Rx) \\
&= \sum_{j,l,m} R_{jk}\delta_{ml}\partial_m\partial_l u_j(Rx) = \sum_j R_{jk}\Delta u_j(Rx).
\end{aligned}$$

Thus

$$\Delta u(Rx) = R^T \Delta u(Rx),$$

as needed. Finally

$$(u \cdot \Delta u)(Rx) = u(Rx) \cdot \Delta u(Rx) = (Ru(x)) \cdot (R\Delta u(x)) = u(x) \cdot \Delta u(x) = (u \cdot \Delta u)(x).$$

Chapter 3

Blow-up on a Cantor set

In this chapter we will prove Theorem 2.2. Namely, given $\xi \in (0,1)$ we will construct a weak solution u to the Navier–Stokes inequality that, at each time instant, is smooth and compactly supported, and such that its singular set has Hausdorff dimension greater than or equal to ξ. In other words, the difference between Theorem 2.2 and Theorem 2.1 (which we proved in the previous chapter) is the size of the singular set. In the case of Theorem 2.1 the singular set is a point $\{x_0\} \times \{T_0\}$ and in the case of Theorem 2.2 it will be a set $S \times \{T_0\}$, where S is a Cantor set with $d_H(S) \in [\xi, 1]$ for given $\xi \in (0,1)$. We will also see how Theorem 2.2 can be obtained by sharpening the proof of Theorem 2.1 (which we shall refer to by writing "previously") as intuitively sketched on Fig. 3.1.

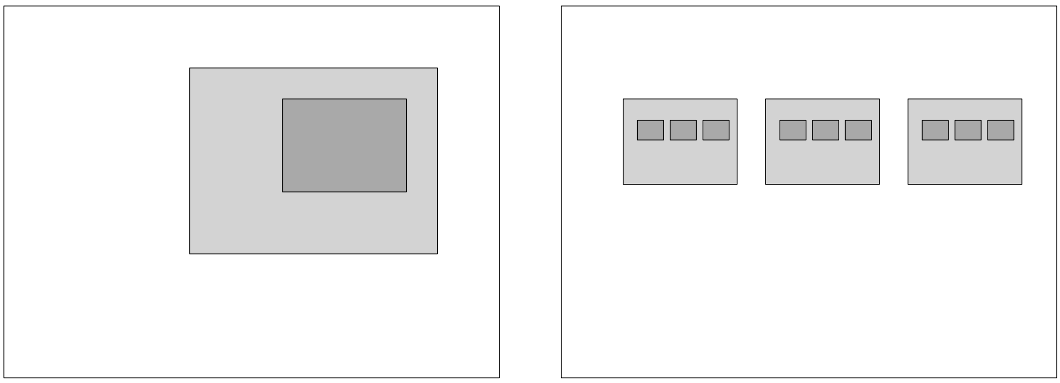

Figure 3.1: The (intuitive) difference in constructing solutions to Theorem 2.1 (left; cf. Fig. 2.1) and Theorem 2.2 (right).

In other words, the solution $\mathfrak{u}$ (of Theorem 2.2) is obtained by a similar switching procedure as in Section 2.1, except that at every switching the support of $\mathfrak{u}$ shrinks (by a fixed factor) to form M copies of itself ($M \in \mathbb{N}$) and thus form a Cantor set S at the limit $t \to T_0^-$. It is remarkable that such an approach allows enough freedom to make S have Hausdorff dimension arbitrarily close to 1 (from below). Before proceeding to the proof we briefly comment on the construction of such a Cantor set and we introduce some useful notation. We then prove Theorem 2.2 in Section 3.2. Furthermore we prove Theorem 2.3 and Theorem 2.4 in Section 3.3.

3.1 Constructing a Cantor set

The problem of constructing Cantor sets is usually demonstrated in a one-dimensional setting using intervals, as in the following proposition.

Proposition 3.1. *Let $I \subset \mathbb{R}$ be an interval and let $\tau \in (0,1)$, $M \in \mathbb{N}$ be such that $\tau M < 1$. Let $C_0 := I$ and consider the iteration in which in the j-th step ($j \geq 1$) the set C_j is obtained by replacing each interval J contained in the set C_{j-1} by M equidistant copies of τJ contained in J, see for example Fig. 3.2. Then the limiting object*

$$C := \bigcap_{j \geq 0} C_j$$

is a Cantor set whose Hausdorff dimension equals $-\log M / \log \tau$.

See Example 4.5 in Falconer (2014) for a proof. Thus if $\tau \in (0,1)$, $M \in \mathbb{N}$ satisfy

$$\tau^\xi M \geq 1 \quad \text{for some } \xi \in (0,1),$$

we obtain a Cantor set C with

$$d_H(C) \geq \xi. \tag{3.1}$$

Note that both the above inequality and the constraint $\tau M < 1$ (which is necessary for the iteration described in the proposition above, see also Fig. 3.2) can be satisfied only for $\xi < 1$. In the remainder of this section we extend the result from the proposition above to the three-dimensional setting.

Let $G \subset \mathbb{R}^3$ be a compact set. We will later take $G := R(\overline{U}_1 \cup \overline{U}_2)$ (as in the case of Theorem 2.1), and so for convenience suppose further that $G = G_1 \cup G_2$ for some disjoint compact sets $G_1, G_2 \subset \mathbb{R}^3$, and such that $G_2 = R(\overline{U_2})$ for some open and connected $U_2 \Subset P$. Let $\tau \in (0,1)$, $M \in \mathbb{N}$, $z = (z_1, z_2, 0) \in G_2$, $X > 0$ be such that

$$\tau^\xi M \geq 1, \qquad \tau M < 1 \tag{3.2}$$

and

$$\{\Gamma_n(G)\}_{n=1,\ldots,M} \text{ is a family of pairwise disjoint subsets of } G_2, \tag{3.3}$$

where

$$\Gamma_n(x) \coloneqq \tau x + z + (n-1)(X,0,0).$$

Equivalently,

$$\Gamma_n(x_1,x_2,x_3) = (\beta_n(x_1),\gamma(x_2),\tau x_3), \tag{3.4}$$

where

$$\begin{cases} \beta_n(x) \coloneqq \tau x + z_1 + (n-1)X, \\ \gamma(x) \coloneqq \tau x + z_2, \end{cases} \qquad x \in \mathbb{R}, n = 1,\ldots,M.$$

Now for $j \geq 1$ let

$$M(j) \coloneqq \{m = (m_1,\ldots,m_j) \colon m_1,\ldots,m_j \in \{1,\ldots,M\}\}$$

denote the set of multi-indices m. Note that in particular $M(1) = \{1,\ldots,M\}$. Informally speaking, each multiindex $m \in M(j)$ plays the role of a "coordinate" which let us identify any component of the set obtained in the j-th step of the construction of the Cantor set. Namely, letting

$$\pi_m \coloneqq \beta_{m_1} \circ \ldots \circ \beta_{m_j}, \qquad m \in M(j),$$

that is

$$\pi_m(x) = \tau^j x + z_1 \frac{1-\tau^j}{1-\tau} + X \sum_{k=1}^{j} \tau^{k-1}(m_k - 1), \quad x \in \mathbb{R} \tag{3.5}$$

we see that the set C_j obtained in the j-th step of the construction of the Cantor set C (from the proposition above) can be expressed simply as

$$C_j \coloneqq \bigcup_{m \in M(j)} \pi_m(I),$$

see Fig. 3.2. Moreover, each $\pi_m(I)$ can be identified by, roughly speaking, first choosing m_1-th subinterval, then m_2-th subinterval, ... , up to m_j-th interval, where $m = (m_1,\ldots,m_j)$. This is demonstrated in Fig. 3.2 in the case when $m = (1,2) \in M(2)$.

In order to proceed with our construction of a Cantor set in three dimensions let

$$\Gamma_m(x_1,x_2,x_3) \coloneqq \left(\pi_m(x_1),\gamma^j(x_2),\tau^j x_3\right).$$

Note that such a definition reduces to (3.4) in the case $j = 1$. If $j = 0$ then let $M(0)$ consist of only one element m_0 and let $\pi_{m_0} \coloneqq \mathrm{id}$. Moreover, if $m \in M(j)$ and

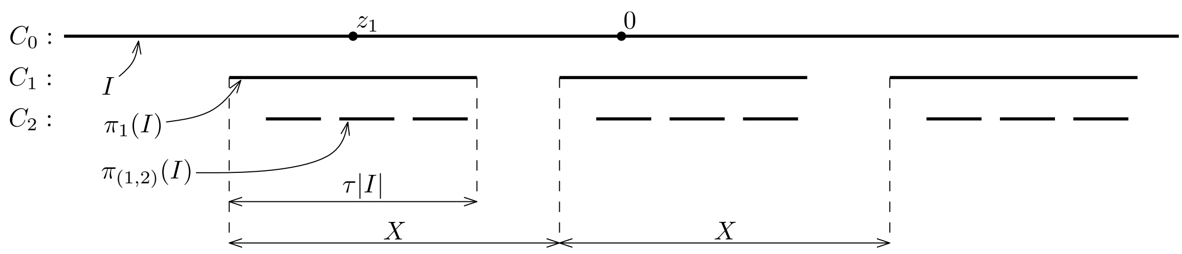

Figure 3.2: A construction of a Cantor set C on a line (here $M = 3$, $j = 0, 1, 2$).

$\overline{m} \in M(j-1)$ is its sub-multiindex, that is $\overline{m} = (m_1, \ldots, m_{j-1})$ ($\overline{m} = m_0$ if $j = 1$), then (3.3) gives

$$\Gamma_m(G) = \Gamma_{\overline{m}}(\Gamma_{m_j}(G)) \subset \Gamma_{\overline{m}}(G_2), \tag{3.6}$$

which is a three-dimensional equivalent of the relation $\pi_m(I) \subset \pi_{\overline{m}}(I)$ (see Fig. 3.2). The above inclusion and (3.3) gives that

$$\Gamma_m(G) \cap \Gamma_{\widetilde{m}}(G) = \emptyset \qquad \text{for } m, \widetilde{m} \in M(j),\ j \geq 1, \text{ with } m \neq \widetilde{m}. \tag{3.7}$$

Another consequence of (3.6) is that the family of sets

$$\left\{ \bigcup_{m \in M(j)} \Gamma_m(G) \right\}_j \quad \text{decreases as } j \text{ increases.} \tag{3.8}$$

Moreover, given j, each of the sets $\Gamma_m(G)$, $m \in M(j)$, is separated from the rest by at least $\tau^{j-1}\zeta$, where $\zeta > 0$ is the distance between $\Gamma_n(G)$ and $\Gamma_{n+1}(G)$, $n = 1, \ldots, M-1$ (recall (3.3)).

Taking the intersection in j we obtain

$$S \coloneqq \bigcap_{j \geq 0} \bigcup_{m \in M(j)} \Gamma_m(G), \tag{3.9}$$

and we now show that

$$\xi \leq d_H(S) \leq 1. \tag{3.10}$$

Noting that S is a subset of a line, the upper bound is trivial. As for the lower bound note that

$$S \supset \bigcap_{j \geq 0} \bigcup_{m \in M(j)} \Gamma_m(G_2) =: S'.$$

Thus, letting $I \subset \mathbb{R}$ be the orthogonal projection of G_2 onto the x_1 axis, we see that I is an interval (since U_2 is connected). Thus the orthogonal projection of S' onto the

x_1 axis is

$$\bigcap_{j\geq 0}\bigcup_{m\in M(j)} \pi_m(I) = C,$$

where C is as in the proposition above. Thus, since the orthogonal projection onto the x_1 axis is a Lipschitz map, we obtain $d_H(S') \geq d_H(C)$ (as a property of Hausdorff dimension, see, for example, Proposition 3.3 in Falconer (2014)). Consequently

$$d_H(S) \geq d_H(S') \geq d_H(C) \geq \xi,$$

as required (recall (3.1) for the last inequality).

3.2 Proof of Theorem 2.2

As in the proof of Theorem 2.1, the proof is based on a geometric arrangement. Here we will need a certain sharper geometric arrangement as follows.

By the *geometric arrangement* (for Theorem 2.2) we mean a pair open sets $U_1, U_2 \Subset P$ together with the corresponding structures (v_1, f_1, ϕ_1), (v_2, f_2, ϕ_2) such that $\overline{U_1} \cap \overline{U_2} = \emptyset$ and, for some $T > 0$, $X > 0$, $\tau \in (0,1)$, $z = (z_1, z_2, 0) \in \mathbb{R}^3$, $M \in \mathbb{N}$, (3.2) and (3.3) hold with

$$G := R\left(\overline{U_1} \cup \overline{U_2}\right),$$

$$f_2^2 + Tv_2 \cdot F[v_1, f_1] > |v_2|^2 \quad \text{in } U_2 \tag{3.11}$$

and

$$f_2^2(y_n) + Tv_2(y_n) \cdot F[v_1, f_1](y_n) > \tau^{-2}\left(f_1(R^{-1}x) + f_2(R^{-1}x)\right)^2 \tag{3.12}$$

for all $x \in G$ and $n = 1, \ldots, M$, where

$$y_n = R^{-1}(\Gamma_n(x)) \tag{3.13}$$

and Γ_n is as in (3.4).

The difference, as compared to the previous geometric arrangement (see Section 2.3) is (3.3) and (3.12), which we now require for all $n = 1, \ldots, M$ (rather than only for $n = 1$, which was the case previously). This reflects the fact that at each switching time we expect the support of $\mathfrak{u}$ to form M copies of itself (rather than one copy, which was the case in Theorem 2.1). Except for this, the inequalities (3.2) specify the relation between τ and M that needs to be satisfied in order to obtain blow-up on a Cantor set with Hausdorff dimension at least ξ. In fact, the previous geometric arrangement is recovered if one takes $\xi = 0$, $M = 1$. We now show how Theorem 2.2 follows (given the geometric arrangement) in a similar way as discussed in Section 2.3, except for a subtle change in the construction of the vector field $u^{(j)}$ (recall the previous construction (2.9)).

To this end, as in Section 2.3, let $\theta > 0$ be sufficiently small such that

$$f_2^2(y_n) + Tv_2(y_n) \cdot F[v_1, f_1](y_n) > \tau^{-2}\left(f_1(R^{-1}x) + f_2(R^{-1}x)\right)^2 + 2\theta \tag{3.14}$$

for $x \in G$, $n = 1, \ldots, M$ (by (3.12)), and set

$$h_t \coloneqq h_{1,t} + h_{2,t}, \tag{3.15}$$

where h_1, h_2 are given by (2.61), (2.62), that is

$$\begin{aligned} h_{1,t}^2 &\coloneqq f_1^2 - 2t\delta\phi_1, \\ h_{2,t}^2 &\coloneqq f_2^2 - 2t\delta\phi_2 + \int_0^t v_2 \cdot F[v_1, h_{1,s}]\,\mathrm{d}s. \end{aligned} \tag{3.16}$$

As in Lemma 2.10, let $\delta > 0$ be sufficiently small so that $h_1, h_2 \in C^\infty(P \times (-\delta, T + \delta); [0, \infty))$,

$$(v_i, h_{i,t}, \phi_i) \text{ is a structure on } U_i \qquad \text{for } t \in (-\delta, T+\delta), i = 1, 2,$$

and

$$h_{2,T}^2(y_n) > \tau^{-2}\left(f_1(R^{-1}x) + f_2(R^{-1}x)\right)^2 + \theta \tag{3.17}$$

for $x \in G$, $n = 1, \ldots, M$. Here only the last inequality differs from the corresponding property (2.64); note however that this is a consequence of (3.14), as previously (2.64) was a consequence of (2.59).

Let $\nu_0 > 0$ be as in (2.68). As in Section 2.1, in order to obtain a solution $\mathfrak{u}$ we want to find $\eta_j > 0$ and a velocity field $u^{(j)}$. However, in contrast to the arguments from Section 2.1, the velocity field $u^{(j)}$ will not be obtained by rescaling a single vector field u (recall (2.9)), which we have pointed out above. In fact, for each j we expect $u^{(j)}$ to consist of M^j disjointly supported vector fields (recall the comments preceding

Section 3.1). A naive idea of constructing $u^{(j)}$ would be to consider M^j rescaled copies of u, that is the vector field

$$\widetilde{u}^{(j)}(x,t) \coloneqq \tau^{-j} \sum_{m\in M(j)} u(\Gamma_m^{-1}(x), \tau^{-2j}(t-t_j)), \qquad j \geq 0.$$

For such vector field

$$\operatorname{supp} \widetilde{u}^{(j)}(t) = \bigcup_{m\in M(j)} \Gamma_m(G), \qquad t\in [t_j, t_{j+1}], j\geq 0,$$

which shrinks to the Cantor set S as $j\to\infty$ (recall (3.9)), as expected. However, the observation that the pressure function does not have a local character (that is the pressure function corresponding to a compactly supported vector field does not have compact support, recall (1.19)) suggests that $\widetilde{u}$ has little chance to satisfy the local energy inequality (1.17). Instead, one needs to make use of the following proposition, which is a generalisation of the previous result (Proposition 2.11) and which utilises the fact that the sets of $\Gamma_m(G)$, $m\in M(j)$, are sufficiently far away from each other (that is $X>0$ is large enough) to ensure that the mutual influence of the pressure functions corresponding to the vector fields supported in $\Gamma_m(G)$, $m\in M(j)$, is very small.

Proposition 3.2. *Let $j\geq 0$ and*

$$h_t^{(j)}(x_1,x_2) \coloneqq \sum_{m\in M(j)} h_t(\pi_m^{-1}(\tau^j x_1), x_2),$$

where h_t is given by (3.15), $t\in(-\delta, T+\delta)$. There exists $\varrho_j>0$ and a vector field $v^{(j)}\in C^\infty\left(\mathbb{R}^3\times(-\varrho_j, T+\varrho_j);\mathbb{R}^3\right)$ such that

(i) $\operatorname{div} v^{(j)}(t)=0$ *and* $\operatorname{supp} v^{(j)}(t) = R\left(\operatorname{supp} h_t^{(j)}\right)$, $t\in(-\varrho_j, T+\varrho_j)$,

(ii) for all $x\in\mathbb{R}^3$, $t\in[0,T]$

$$\left|v^{(j)}(x,0)\right| = h_0^{(j)}\left(R^{-1}x\right), \qquad \left|\left|v^{(j)}(x,t)\right|^2 - h_t^{(j)}\left(R^{-1}x\right)^2\right| < \theta,$$

(iii) the Navier–Stokes inequality

$$\partial_t \left|v^{(j)}\right|^2 \leq -v^{(j)}\cdot\nabla\left(\left|v^{(j)}\right|^2 + 2\overline{p}^{(j)}\right) + 2\nu\, v^{(j)}\cdot\Delta v^{(j)}$$

is satisfied in $\mathbb{R}^3\times[0,T]$ for every $\nu\in[0,\nu_0]$, and *(iv)* $\|v^{(j)}(t)\|_{L^\infty} \le \mathcal{C}$ *for* $t \in [0,T]$ *and*

$$\int_0^T \|\nabla v^{(j)}(t)\|_{L^2}^2 \mathrm{d}t, \int_0^T \|v^{(j)}(t)\|_{L^3}^3 \mathrm{d}t, \int_0^T \|v^{(j)}(t)\overline{p}^{(j)}(t)\|_{L^2}^2 \mathrm{d}t \le \mathcal{C},$$

for some constant $\mathcal{C} > 0$ *which is independent of* j*, where* $\overline{p}^{(j)}$ *is the pressure function corresponding to* $v^{(j)}$*.*

Given the claim of the proposition above (which we prove in Section 3.2.1 below) we let

$$u^{(j)}(x_1,x_2,x_3,t) \coloneqq \tau^{-j} v^{(j)}(\tau^{-j}x_1, \gamma^{-j}(x_2), \tau^{-j}x_3, \tau^{-2j}(t-t_j)), \tag{3.18}$$

where $t_0 \coloneqq 0$ and $t_j \coloneqq T\sum_{k=0}^{j-1}\tau^{2k}$, as previously. Then, as in Section 2.3 (cf. Proposition 2.11), claims (i), (iii) imply that $u^{(j)} \in C^\infty(\mathbb{R}^3 \times [t_j, t_{j+1}]; \mathbb{R}^3)$ is divergence free and satisfies the Navier–Stokes inequality

$$\partial_t \left|u^{(j)}\right|^2 \le -u^{(j)} \cdot \nabla \left(\left|u^{(j)}\right|^2 + 2p^{(j)} \right) + 2\nu\, u^{(j)} \cdot \Delta u^{(j)} \tag{3.19}$$

in $\mathbb{R}^3 \times [t_j, t_{j+1}]$ for all $\nu \in [0, \nu_0]$, where $p^{(j)}$ is the pressure function corresponding to $u^{(j)}$ (recall that $C^\infty(\mathbb{R}^3 \times [a,b]; \mathbb{R}^3)$ denotes the space of vector functions that are infinitely differentiable on $\mathbb{R}^3 \times (a-\eta, b+\eta)$ for some $\eta > 0$). Moreover (in contrast to the previous relation (2.10)) (i) gives

$$\operatorname{supp} u^{(j)}(t) = \bigcup_{m \in M(j)} \Gamma_m(G), \qquad t \in [t_j, t_{j+1}], \tag{3.20}$$

and (ii) gives

$$\begin{aligned} \left|u^{(j)}(x,t_j)\right| &= \tau^{-j} \sum_{m \in M(j)} h_0(R^{-1}(\Gamma_m^{-1}(x))), \\ \left|u^{(j)}(x,t_{j+1})\right|^2 &> \tau^{-2j} \sum_{m \in M(j)} h_T(R^{-1}(\Gamma_m^{-1}(x)))^2 - \tau^{-2j}\theta, \quad x \in \mathbb{R}^3, \end{aligned} \tag{3.21}$$

which can be used to show that

$$\left|u^{(j)}(x,t_j)\right| \le \left|u^{(j-1)}(x,t_j)\right|, \qquad x \in \mathbb{R}^3, j \ge 1, \tag{3.22}$$

in a similar way as (2.11). Indeed, in order to see this note that this inequality is nontrivial only for $x \in \bigcup_{m \in M(j)} \Gamma_m(G)$, and so let $j \ge 1$, $m \in M(j)$ be such that $x = \Gamma_m(y)$ for some $y \in G$. Then, in the light of (3.7), we see that $\Gamma_{\widetilde{m}}^{-1}(x) \notin G$ for any $\widetilde{m} \in M(j-1)$, $\widetilde{m} \ne m$, and so the first line of (3.21) becomes simply

$$\left|u^{(j)}(x,t_j)\right| = \tau^{-j} h_0(R^{-1}(y)). \tag{3.23}$$

Furthermore, letting $\overline{m} \in M(j-1)$ be the sub-multiindex of m, that is $m = (\overline{m}, m_j)$ for some $m_j \in \{1, \ldots, M\}$, we see that

$$x = \Gamma_{\overline{m}} \left(\Gamma_{m_j}(y) \right).$$

This means that, at $(j-1)$-th step (that is for $t \in [t_{j-1}, t_j]$) x was an element of $\Gamma_{\overline{m}}(G)$ (and at time t_j this component of $\operatorname{supp} u^{(j-1)}$ will divide into M disjoint copies, $\{\Gamma_{\overline{m},n}(G)\}_{n=1,\ldots,M}$, which will become M out of M^j components of $\operatorname{supp} u^{(j)}$ (see (3.20)); and among these copies x belongs to $\Gamma_m(G)$). Therefore, as in (3.23) above we see that the second line of (3.21) is simply

$$\begin{aligned}\left| u^{(j-1)}(x, t_j) \right|^2 &> \tau^{-2(j-1)} h_T(R^{-1}(\Gamma_{\overline{m}}^{-1}(x)))^2 - \tau^{-2(j-1)}\theta \\ &= \tau^{-2(j-1)} h_T(R^{-1}(\Gamma_{m_j}(y)))^2 - \tau^{-2(j-1)}\theta. \\ &= \tau^{-2(j-1)} h_{2,T}(R^{-1}(\Gamma_{m_j}(y)))^2 - \tau^{-2(j-1)}\theta,\end{aligned}$$

where, in the last inequality, we used the fact that $h_{1,T}(R^{-1}(\Gamma_{m_j}(y))) = 0$ (recall (3.3) gives $R^{-1}(\Gamma_{m_j}(y)) \in \overline{U_2}$). From this and (3.23) we obtain (3.22) by an easy calculation,

$$\begin{aligned}\left| u^{(j-1)}(x, t_j) \right|^2 &> \tau^{-2(j-1)} h_{2,T}(R^{-1}(\Gamma_{m_j}(y)))^2 - \tau^{-2(j-1)}\theta \\ &> \tau^{-2j} \left(f_1(R^{-1}(y)) + f_2(R^{-1}(y)) \right)^2 \\ &= \tau^{-2j} h_0(R^{-1}(y))^2 \\ &= \left| u^{(j)}(x, t_j) \right|^2,\end{aligned}$$

where we used (3.17) in the second inequality.

Hence, letting

$$\mathfrak{u}(t) := \begin{cases} u^{(j)}(t) & \text{if } t \in [t_j, t_{j+1}) \text{ for some } j \geq 0, \\ 0 & \text{if } t \geq T_0, \end{cases} \tag{3.24}$$

where $T_0 := \lim_{j\to\infty} t_j = T/(1-\tau^2)$ (as previously), we obtain a solution to Theorem 2.2. Indeed, that $\mathfrak{u}$ is a weak solution to the NSI follows as in the case of Theorem 2.1 (note that, in order to obtain the required regularity $\sup_{t>0} \|\mathfrak{u}\| < \infty$, $\nabla u \in L^2(\mathbb{R}^3 \times (0,\infty))$ it suffices to replace "τ" by "$M\tau$" in the calculations (2.13), (2.14)).

Furthermore the singular set of $\mathfrak{u}$ is

$$S \times \{T_0\} = \left(\bigcap_{j\geq 0} \bigcup_{m \in M(j)} \Gamma_m(G) \right) \times \{T_0\}.$$

Indeed, (3.20) shows that the support of $\mathfrak{u}(t)$ consists of M^j components for $t \in [t_j, t_{j+1})$ and that it shrinks to the Cantor set S as $t \to T_0^-$. That $\mathfrak{u}$ is unbounded in any neighbourhood V of any point $(y, T_0) \in S \times \{T_0\}$ follows from Proposition 3.2 (ii) and (3.18), which show that the magnitude of $\mathfrak{u}$ grows uniformly on each component of its support; in other words given any positive number $\mathcal{N}$ let $x \in G$ be any point such that $h_0(R^{-1}(x)) > 0$ and let $j \geq 0$, $m \in M(j)$ be such that

$$\Gamma_m(G) \times [t_j, T_0] \subset V \quad \text{and} \quad \tau^{-j} \geq \mathcal{N}/h_0(R^{-1}(x)).$$

Then

$$|\mathfrak{u}(\Gamma_m(x), t_j)| = \tau^{-j} h_0^{(j)}(R^{-1}(\tau^{-j}\pi_m(x_1), x_2, x_3)) = \tau^{-j} h_0(R^{-1}(x)) \geq \mathcal{N}.$$

Thus $S \times \{T_0\}$ is a singular set of $\mathfrak{u}$ whose Hausdorff dimension is greater than ξ due to (3.10).

3.2.1 Proof of Proposition 3.2

Here we prove Proposition 3.2, which concludes the proof of Theorem 2.2 (given the geometric arrangement, which we present in Section 3.2.3). Fix $j \geq 0$. We will write, for brevity, $v = v^{(j)}$, $\overline{p} = \overline{p}^{(j)}$.

Step 1. Renumber the functions $h_i(\pi_m^{-1}(\tau^j x_1), x_2, t)$.

For brevity let $\mathfrak{M} := M^j$, identify each multiindex $m \in M(j)$ with an integer $\mathfrak{m} \in \{1, \ldots, \mathfrak{M}\}$, and let

$$h_i^{\mathfrak{m}}(x_1, x_2, t) := h_i(\pi_m^{-1}(\tau^j x_1), x_2, t), \qquad i = 1, 2, \tag{3.25}$$

and

$$\begin{cases} f_i^{\mathfrak{m}}(x_1, x_2) := f_i(\pi_m^{-1}(\tau^j x_1), x_2), \\ v_i^{\mathfrak{m}}(x_1, x_2) := v_i(\pi_m^{-1}(\tau^j x_1), x_2), \\ \phi_i^{\mathfrak{m}}(x_1, x_2) := \phi_i(\pi_m^{-1}(\tau^j x_1), x_2), \\ U_i^{\mathfrak{m}} := \{(x_1, x_2) \colon (\pi_m^{-1}(\tau^j x_1), x_2) \in U_i\} \end{cases} \qquad i = 1, 2.$$

Then $h_1^{\mathfrak{m}}, h_2^{\mathfrak{m}} \in C^\infty(P \times (-\delta, T+\delta); [0, \infty))$,

$$(v_i^{\mathfrak{m}}, f_i^{\mathfrak{m}}, \phi_i^{\mathfrak{m}}) \text{ and } (v_i^{\mathfrak{m}}, h_{i,t}^{\mathfrak{m}}, \phi_i^{\mathfrak{m}}) \text{ are structures on } U_i^{\mathfrak{m}} \qquad \text{for } t \in (-\delta, T+\delta), i = 1, 2,$$

and

$$h_t^{(j)} = \sum_{\mathfrak{m}=1}^{\mathfrak{M}} (h_1^{\mathfrak{m}} + h_2^{\mathfrak{m}}) .$$

Moreover,

$$\operatorname{supp}(h_{1,t}^{\mathfrak{m}} + h_{2,t}^{\mathfrak{m}}) = \overline{U_1^{\mathfrak{m}}} \cup \overline{U_2^{\mathfrak{m}}} =: K^{\mathfrak{m}}$$

and the sets $K^{\mathfrak{m}}$ are pairwise disjoint translates of $\overline{U_1} \cup \overline{U_2}$ in the x_1 direction, such that the distance between any $K^{\mathfrak{m}}$ and $K^{\mathfrak{n}}$ for $\mathfrak{m}, \mathfrak{n} \in \{1, \ldots, \mathfrak{M}\}$, $\mathfrak{n} \neq \mathfrak{m}$, is at least $\tau^{-1}\zeta$ (just as each element of the union $\bigcup_{m \in M(j)} \Gamma_m(G)$ is separated from the rest by at least $\tau^{j-1}\zeta$, see the comments preceding (3.10)). Furthermore, we can assume that the bijection $m \longleftrightarrow \mathfrak{m}$ is such that $K^{\mathfrak{m}+1}$ is a positive translate of $K^{\mathfrak{m}}$ in the x_1 direction, that is

$$K^{\mathfrak{m}+1} = K^{\mathfrak{m}} + (a_{\mathfrak{m}}, 0) \quad \text{for some } a_{\mathfrak{m}} > 0, \;\; \mathfrak{m} = 1, \ldots, \mathfrak{M} - 1.$$

For such a bijection

$$\operatorname{dist}(K^{\mathfrak{n}}, K^{\mathfrak{m}}) \geq |\mathfrak{n} - \mathfrak{m}| \tau^{-1}\zeta, \qquad \mathfrak{n}, \mathfrak{m} = 1, \ldots, \mathfrak{M}. \tag{3.26}$$

Step 2. Introduce modifications $q_{i,t}^{\mathfrak{m},k}$ of the functions $h_{i,t}^{\mathfrak{m}}$.

Let

$$\left(q_{i,t}^{\mathfrak{m},k}\right)^2 := \left(h_{i,0}^{\mathfrak{m}}\right)^2 - 2t\delta\phi_i^{\mathfrak{m}} - \int_0^t a_i^{\mathfrak{m},k}(s) v_i^{\mathfrak{m}} \cdot \left(\nabla (h_{i,s}^{\mathfrak{m}})^2 + 2 \sum_{l=1,2} \sum_{\mathfrak{n}=1}^{\mathfrak{M}} \nabla p[a_l^{\mathfrak{n},k}(s) v_l^{\mathfrak{n}}, h_{l,s}^{\mathfrak{n}}] \right) \mathrm{d}s, \tag{3.27}$$

$i = 1, 2$, $k \in \mathbb{N}$, $\mathfrak{m} = 1, \ldots, \mathfrak{M}$, where $a_i^{\mathfrak{m},k} \in C^\infty(\mathbb{R}; [-1, 1])$, $i = 1, 2$, $m = 1, \ldots, \mathcal{M}$ are oscillatory functions constructed below. Observe that this is a natural extension of the idea from Section 2.3.1 to the case of $\mathfrak{M}$ pairs $U_1^{\mathfrak{m}}, U_2^{\mathfrak{m}}$ (rather than a single pair U_1, U_2, which was the case previously). Note that such a definition gives

$$\partial_t \left(q_{i,t}^{\mathfrak{m},k}\right)^2 = -2\delta\phi_i^{\mathfrak{m}} + a_i^{\mathfrak{m},k}(t) v_i^{\mathfrak{m}} \cdot \left(\nabla (h_{i,t}^{\mathfrak{m}})^2 + 2 \sum_{l=1,2} \sum_{\mathfrak{n}=1}^{\mathfrak{M}} \nabla p[a_l^{\mathfrak{n},k}(t) v_l^{\mathfrak{n}}, h_{l,t}^{\mathfrak{n}}] \right). \tag{3.28}$$

As in (2.75), we will construct the oscillatory processes $a_1^{\mathfrak{m},k}, a_2^{\mathfrak{m},k} \in C^\infty(\mathbb{R}; [-1, 1])$ in such a way that

$$\begin{cases} q_{i,t}^{\mathfrak{m},k} \to h_{i,t}^{\mathfrak{m}} \\ \text{and} \\ D^l q_{i,t}^{\mathfrak{m},k} \to D^l h_{i,t}^{\mathfrak{m}} \end{cases} \qquad \text{uniformly in } P \times [0, T], i = 1, 2, \mathfrak{m} \in \{1, \ldots, \mathfrak{M}\} \tag{3.29}$$

for each $l \geq 1$. Then, as in Section 2.3.1, we obtain that for sufficiently large k

$$(a_i^{\mathfrak{m},k}(t)v_i^{\mathfrak{m}}, q_{i,t}^{\mathfrak{m},k}, \phi_i^{\mathfrak{m}}) \text{ is a structure on } U_i^{\mathfrak{m}} \text{ for } t \in (-\delta_k, T+\delta_k), i = 1,2,$$

and that

$$q_i^{\mathfrak{m},k} \in C^\infty(P \times (-\delta_k, T+\delta_k); [0,\infty)).$$

Finally let

$$v(x,t) := \sum_{\mathfrak{m}=1}^{\mathfrak{M}} \left(u[a_1^{\mathfrak{m},k}(t)v_1^{\mathfrak{m}}, q_{1,t}^{\mathfrak{m},k}](x) + u[a_2^{\mathfrak{m},k}(t)v_2^{\mathfrak{m}}, q_{2,t}^{\mathfrak{m},k}](x) \right), \tag{3.30}$$

Step 3. Verify that v satisfies the claims of the theorem.

We will now show that (given the existence of the oscillatory processes $a_1^{\mathfrak{m},k}, a_2^{\mathfrak{m},k}$, which we show in Section 3.2.2) the function (3.30) is a solution to Proposition 3.2.

Claim (i) is trivial, similarly as claim (ii) given k large enough such that

$$\left| (q_{i,t}^{\mathfrak{m},k})^2 - (h_{i,t}^{\mathfrak{m}})^2 \right| \leq \theta/2 \qquad \text{in } P, t \in [0,T], i = 1,2.$$

As for claim (iii), the Navier–Stokes inequality, note that since $v^{(j)}$ is rotationally invariant, it is equivalent to

$$\partial_t |v(x,0,t)|^2 \leq -v(x,0,t) \cdot \nabla \left(|v(x,0,t)|^2 + 2\overline{p}(x,0,t) \right) + 2\nu\, v(x,0,t) \cdot \Delta v(x,0,t),$$

where $\nu \in [0,\nu_0]$, $x \in P$, $t \in [0,T]$ and $\overline{p}$ is the pressure function corresponding to v, that is

$$\overline{p}(t) = \sum_{\mathfrak{m}=1}^{\mathfrak{M}} \left(p^*[a_1^{\mathfrak{m},k}(t)v_1^{\mathfrak{m}}, q_{1,t}^{\mathfrak{m},k}] + p^*[a_2^{\mathfrak{m},k}(t)v_2^{\mathfrak{m}}, q_{2,t}^{\mathfrak{m},k}] \right) \tag{3.31}$$

(recall (2.34) and Lemma 2.6 (iii)), which in particular means that

$$\overline{p}(x,0,t) = \sum_{\mathfrak{m}=1}^{\mathfrak{M}} \left(p[a_1^{\mathfrak{m},k}(t)v_1^{\mathfrak{m}}, q_{1,t}^{\mathfrak{m},k}](x) + p[a_2^{\mathfrak{m},k}(t)v_2^{\mathfrak{m}}, q_{2,t}^{\mathfrak{m},k}](x) \right), \quad x \in P.$$

As in Section 2.3.2 we fix $x \in P$, $t \in [0,T]$ and consider two cases.

Case 1. $\phi_1^{\mathfrak{m}}(x) + \phi_2^{\mathfrak{m}}(x) < 1$ for all $\mathfrak{m} \in \{1,\ldots,\mathfrak{M}\}$. For such x we have $v_1^{\mathfrak{m}}(x) = v_2^{\mathfrak{m}}(x) = 0$ and the Navier–Stokes inequality follows trivially for all k by writing

$$\begin{aligned}
\partial_t |v(x,0,t)|^2 &= \sum_{\mathfrak{m}=1}^{\mathfrak{M}} \left(\partial_t q_{1,t}^{\mathfrak{m},k}(x)^2 + \partial_t q_{2,t}^{\mathfrak{m},k}(x)^2 \right) \\
&= -2\delta \sum_{\mathfrak{m}=1}^{\mathfrak{M}} (\phi_1^{\mathfrak{m}}(x) + \phi_2^{\mathfrak{m}}(x)) \\
&\leq 0 \\
&\leq -v(x,0,t) \cdot \nabla \left(|v(x,0,t)|^2 + 2\overline{p}(x,0,t) \right) + 2\nu\, v(x,0,t) \cdot \Delta v(x,0,t),
\end{aligned}$$

where we used (2.43) and (2.44) in the last step.

Case 2. $\phi_1^{\mathfrak{m}}(x)+\phi_2^{\mathfrak{m}}(x)=1$ for some $\mathfrak{m}\in\{1,\ldots,\mathfrak{M}\}$. In this case we need to use the convergence (3.29) with k sufficiently large such that

$$|v_i^{\mathfrak{m}}|\left(\left|\nabla(q_{i,t}^{\mathfrak{m},k})^2-\nabla(h_{i,t}^{\mathfrak{m}})^2\right|+2\sum_{\mathfrak{n}=1}^{\mathfrak{M}}\sum_{l=1,2}\left|\nabla p[a_l^{\mathfrak{n},k}(t)v_l^{\mathfrak{n}},q_{l,t}^{\mathfrak{n},k}]-\nabla p[a_l^{\mathfrak{n},k}(t)v_l^{\mathfrak{n}},h_{l,t}^{\mathfrak{n}}]\right|\right)\leq\delta/2 \tag{3.32}$$

in P and

$$\begin{aligned}&\nu_0\left|u[a_i^{\mathfrak{m},k}(t)v_i^{\mathfrak{m}},q_{i,t}^{\mathfrak{m},k}]\cdot\Delta u[a_i^{\mathfrak{m},k}(t)v_i^{\mathfrak{m}},q_{i,t}^{\mathfrak{m},k}]\right|\\&\leq\nu_0\left|u[a_i^{\mathfrak{m},k}(t)v_i^{\mathfrak{m}},h_{i,t}^{\mathfrak{m}}]\cdot\Delta u[a_i^{\mathfrak{m},k}(t)v_i^{\mathfrak{m}},h_{i,t}^{\mathfrak{m}}]\right|+\delta/8\leq\delta/4\end{aligned} \tag{3.33}$$

in $\mathbb{R}^3$, for $t\in[0,T]$, $i=1,2$. We obtain

$$\begin{aligned}\partial_t|v(x,0,t)|^2&=\partial_t q_{1,t}^{\mathfrak{m},k}(x)^2+\partial_t q_{2,t}^{\mathfrak{m},k}(x)^2\\&=-2\delta-\left(a_1^{\mathfrak{m},k}(t)v_1^{\mathfrak{m}}(x)+a_2^{\mathfrak{m},k}(t)v_2^{\mathfrak{m}}(x)\right)\cdot\nabla\Bigg(\left(h_{1,t}^{\mathfrak{m}}(x)\right)^2+\left(h_{2,t}^{\mathfrak{m}}(x)\right)^2\\&\qquad+2\sum_{\mathfrak{n}=1}^{\mathfrak{M}}\left(p[a_1^{\mathfrak{n},k}(t)v_1^{\mathfrak{n}},h_{1,t}^{\mathfrak{n}}](x)+p[a_2^{\mathfrak{n},k}(t)v_2^{\mathfrak{n}},h_{2,t}^{\mathfrak{n}}](x)\right)\Bigg)\\&\leq-\delta-\left(a_1^{\mathfrak{m},k}(t)v_1^{\mathfrak{m}}(x)+a_2^{\mathfrak{m},k}(t)v_2^{\mathfrak{m}}(x)\right)\cdot\nabla\Bigg(q_{1,t}^{\mathfrak{m},k}(x)^2+q_{2,t}^{\mathfrak{m},k}(x)^2\\&\qquad+2\sum_{\mathfrak{n}=1}^{\mathfrak{M}}\left(p[a_1^{\mathfrak{n},k}(t)v_1^{\mathfrak{n}},q_{1,t}^{\mathfrak{n},k}](x)+p[a_2^{\mathfrak{n},k}(t)v_2^{\mathfrak{n}},q_{2,t}^{\mathfrak{n},k}](x)\right)\Bigg)\\&=-\delta-v_1(x,0,t)\partial_{x_1}\left(|v(x,0,t)|^2+2\overline{p}(x,0,t)\right)\\&\qquad-v_2(x,0,t)\partial_{x_2}\left(|v(x,0,t)|^2+2\overline{p}(x,0,t)\right),\end{aligned}$$

and so, recalling that $\partial_{x_3}|v(x,0,t)|^2=\partial_{x_3}\overline{p}(x,0,t)=0$ (as a property of rotationally invariant functions, see (2.32) and (2.37)),

$$\begin{aligned}\partial_t|v(x,0,t)|^2&\leq-\delta-v(x,0,t)\cdot\nabla\left(|v(x,0,t)|^2+2\overline{p}(x,0,t)\right)\\&\leq 2\nu\,v(x,0,t)\cdot\Delta v(x,0,t)-v(x,0,t)\cdot\nabla\left(|v(x,0,t)|^2+2\overline{p}(x,0,t)\right)\end{aligned}$$

for all $\nu\in[0,\nu_0]$, where we used (3.33) in the last step.

It remains to verify (iv). For this note that

$$|v(x,t)|=\sum_{\mathfrak{m}=1}^{\mathfrak{M}}\left(|q_{1,t}^{\mathfrak{m},k}(R^{-1}x)+q_{2,t}^{\mathfrak{m},k}(R^{-1}x)|\right)$$

(recall that $\{q_{i,t}^{\mathfrak{m},k}\}_{i=1,2,\,\mathfrak{m}=1,\ldots,\mathfrak{M}}$ have disjoint supports $U_i^{\mathfrak{m}}$, respectively), and thus, in the view of (3.29), for sufficiently large k

$$\begin{aligned} |v(x,t)| &\leq \sum_{\mathfrak{m}=1}^{\mathfrak{M}} \left| h_{1,t}^{\mathfrak{m}}(R^{-1}x) + h_{2,t}^{\mathfrak{m}}(R^{-1}x) \right| + 1 \\ &\leq \sup_{s\in[0,T]} \|h_{1,s} + h_{2,s}\|_{L^\infty} + 1, \end{aligned} \tag{3.34}$$

since the functions $h_{1,t}^{\mathfrak{m}} + h_{2,t}^{\mathfrak{m}}$ have disjoint supports $K^{\mathfrak{m}}$ ($\mathfrak{m} = 1,\ldots,\mathfrak{M}$). Hence, since $\operatorname{supp} v(t) = \bigcup_{\mathfrak{m}=1}^{\mathfrak{M}} R(K^{\mathfrak{m}})$ consists of $\mathfrak{M}$ copies of $R(\overline{U_1} \cup \overline{U_2})$ we obtain, by Hölder's inequality, that

$$\|v(t)\|_{L^2} \leq \mathfrak{M}\mathcal{C}, \qquad t\in[0,T],$$

and

$$\int_0^T \|v(t)\|_{L^3}^3 \mathrm{d}t \leq \mathfrak{M}\mathcal{C}$$

for some $\mathcal{C} > 0$ independent of j (we write $\mathcal{C}$ for a constant, not to be confused with C, which is a constant related to the decay of the pressure function and was fixed above Section 2.4.1). Similarly for sufficiently large k

$$\|q_{1,t}^{\mathfrak{m},k} + q_{2,t}^{\mathfrak{m},k}\|_{W^{1,\infty}} \leq \|h_{1,t}^{\mathfrak{m}} + h_{2,t}^{\mathfrak{m}}\|_{W^{1,\infty}} + 1,$$

and so, applying (2.46), we obtain

$$\begin{aligned} |\nabla v(x,t)| &\leq \sum_{\mathfrak{m}=1}^{\mathfrak{M}} \left| \nabla u[a_1^{\mathfrak{m},k}(t) v_1^{\mathfrak{m}}, q_{1,t}^{\mathfrak{m},k}](x) + \nabla u[a_2^{\mathfrak{m},k}(t) v_2^{\mathfrak{m}}, q_{2,t}^{\mathfrak{m},k}](x) \right| \\ &\leq \max_{\mathfrak{m}\in\{1,\ldots,\mathfrak{M}\}} \mathcal{C}(\|v_1^{\mathfrak{m}} + v_2^{\mathfrak{m}}\|_{W^{1,\infty}}, \|q_{1,t}^{\mathfrak{m},k} + q_{2,t}^{\mathfrak{m},k}\|_{W^{1,\infty}}) \\ &\leq \max_{\mathfrak{m}\in\{1,\ldots,\mathfrak{M}\}, s\in[0,T]} \mathcal{C}(\|v_1^{\mathfrak{m}} + v_2^{\mathfrak{m}}\|_{W^{1,\infty}}, \|h_{1,s}^{\mathfrak{m}} + h_{2,s}^{\mathfrak{m}}\|_{W^{1,\infty}} + 1) \\ &= \max_{s\in[0,T]} \mathcal{C}(\|v_1 + v_2\|_{W^{1,\infty}}, \|h_{1,s} + h_{2,s}\|_{W^{1,\infty}} + 1), \end{aligned} \tag{3.35}$$

and therefore

$$\int_0^T \|\nabla v(t)\|_{L^2}^2 \mathrm{d}t \leq \mathfrak{M}\mathcal{C}$$

for some $\mathcal{C} > 0$ independent of j. In order to obtain a similar bound on the integral $\int_0^T \|v(t)\overline{p}(t)\|_{L^1}\mathrm{d}t$ it is enough to show that $\overline{p}(t)$ is bounded on $\operatorname{supp} v(t) = \bigcup_{\mathfrak{m}=1}^{\mathfrak{M}} R(K^{\mathfrak{m}})$ (by a constant independent of j). For this recall that $\overline{p}(t)$ is given by

$$\overline{p}(x,t) = \int_{\mathbb{R}^3} \sum_{i,j=1}^3 \frac{\partial_i v_j(y,t) \partial_j v_i(y,t)}{4\pi|x-y|} \mathrm{d}y$$

recall (3.31), and fix $x \in K^{\mathfrak{m}}$ for some $\mathfrak{m} \in \{1, \ldots, \mathfrak{M}\}$. Separating the integral $\int_{\mathbb{R}^3}$ into

$$\int_{R(K^{\mathfrak{m}})} + \sum_{\mathfrak{n}=1,\mathfrak{n}\neq\mathfrak{m}}^{\mathfrak{M}} \int_{R(K^{\mathfrak{n}})}$$

we can bound the integral $\int_{R(K^{\mathfrak{m}})}$ using (3.35),

$$\left| \int_{R(K^{\mathfrak{m}})} \sum_{i,j=1}^{3} \frac{\partial_i v_j(y,t) \partial_j v_i(y,t)}{4\pi |x-y|} \mathrm{d}y \right| \le \frac{9}{4\pi} \|\nabla v(t)\|_{L^\infty}^2 \int_{R(K^{\mathfrak{m}})} |x-y|^{-1} \mathrm{d}y \le \mathcal{C}$$

for some $\mathcal{C}$ independent of j. As for the remaining integrals $\sum_{\mathfrak{n}=1,\mathfrak{n}\neq\mathfrak{m}}^{\mathfrak{M}} \int_{R(K^{\mathfrak{n}})}$, integrate by parts in x_j and x_i, and use (3.34) and (3.26) to write

$$\begin{aligned} \left| \sum_{\mathfrak{n}=1,\mathfrak{n}\neq\mathfrak{m}}^{\mathfrak{M}} \int_{R(K^{\mathfrak{n}})} \sum_{i,j=1}^{3} \frac{\partial_i v_j(y,t) \partial_j v_i(y,t)}{4\pi |x-y|} \mathrm{d}y \right| &\le \frac{9}{4\pi} \|v(t)\|_{L^\infty}^2 \sum_{\mathfrak{n}=1,\mathfrak{n}\neq\mathfrak{m}}^{\mathfrak{M}} \int_{R(K^{\mathfrak{n}})} |x-y|^{-3} \mathrm{d}y \\ &\le \mathcal{C} \sum_{\mathfrak{n}=1,\mathfrak{n}\neq\mathfrak{m}}^{\mathfrak{M}} \operatorname{dist}(K^{\mathfrak{m}}, K^{\mathfrak{n}})^{-3} \\ &\le \mathcal{C}\tau^{-1}\delta \sum_{\mathfrak{n}=1,\mathfrak{n}\neq\mathfrak{m}}^{\mathfrak{M}} |\mathfrak{m}-\mathfrak{n}|^{-3} \\ &\le \mathcal{C}\tau^{-1}\delta \sum_{\mathfrak{n}=1}^{\infty} \mathfrak{n}^{-3} \\ &= \mathcal{C} \end{aligned}$$

for some constant $\mathcal{C} > 0$ which does not depend on j (and whose value change from line to line).

3.2.2 The new oscillatory processes

Here we prove the existence of oscillatory processes $a_1^{\mathfrak{m},k}, a_2^{\mathfrak{m},k} \in C^\infty(\mathbb{R}; [-1,1])$ which give the convergence (3.29). The construction of such oscillatory processes is a natural extension of the construction of the processes a_1^k, a_2^k from Section 2.3.3 to the case of $\mathfrak{M}$ pairs $U_1^{\mathfrak{m}}$, $U_2^{\mathfrak{m}}$ (and the corresponding structures, $\mathfrak{m} = 1, \ldots, \mathfrak{M}$). In particular we will use the following sharper version of Theorem 2.12.

Theorem 3.3. *For each $k \ge 1$, $\mathfrak{m} = 1, \ldots, \mathfrak{M}$ there exist a pair of functions $a_1^{\mathfrak{m},k}, a_2^{\mathfrak{m},k} \in$*

$C^\infty(\mathbb{R};[-1,1])$, $i=1,2$, *such that*

$$\begin{aligned}&\int_0^t a_i^{\mathfrak{m},k}(s)\left(G_i^{\mathfrak{m}}(x,s)+\sum_{l=1,2}\sum_{\mathfrak{n}=1}^{\mathfrak{M}} F_{i,l}^{\mathfrak{m},\mathfrak{n}}\left(x,s,a_j^{\mathfrak{n},k}(s)\right)\right)\mathrm{d}s\\ &\xrightarrow{k\to\infty}\begin{cases}\frac{1}{2}\int_0^t\left(F_{2,1}^{\mathfrak{m},\mathfrak{m}}(x,s,1)-F_{2,1}^{\mathfrak{m},\mathfrak{m}}(x,s,0)\right)\mathrm{d}s & i=2,\\ 0 & i=1\end{cases}\end{aligned}\tag{3.36}$$

uniformly in $(x,t)\in P\times[0,T]$, $\mathfrak{m}=1,\ldots,\mathfrak{M}$ *for any bounded and uniformly continuous functions*

$$G_i^{\mathfrak{m}}\colon P\times[0,T]\to\mathbb{R},\qquad F_{i,l}^{\mathfrak{m},\mathfrak{n}}\colon P\times[0,T]\times[-1,1]\to\mathbb{R},$$

$i,l=1,2$, $\mathfrak{m},\mathfrak{n}=1,\ldots,\mathfrak{M}$ *satisfying*

$$F_{i,l}^{\mathfrak{m},\mathfrak{n}}(x,t,-1)=F_{i,l}^{\mathfrak{m},\mathfrak{n}}(x,t,1)\qquad \text{for } x\in P, t\in[0,T],\ i,l=1,2,\ \mathfrak{m},\mathfrak{n}=1,\ldots,\mathfrak{M}.$$

Note that, as in Section 2.3.3, this theorem gives (3.29) simply by taking

$$\begin{aligned}G_i^{\mathfrak{m}}(x,t)&\coloneqq v_i^{\mathfrak{m}}(x)\cdot\nabla(h_i^{\mathfrak{m}}(x,t))^2,\\ F_{i,l}^{\mathfrak{m},\mathfrak{n}}(x,t,a)&\coloneqq 2v_i^{\mathfrak{m}}(x)\cdot\nabla p[av_l^{\mathfrak{n}},h_{l,t}^{\mathfrak{n}}](x)\end{aligned}$$

(recall $F_{i,l}^{\mathfrak{m},\mathfrak{n}}(x,t,-1)=F_{i,l}^{\mathfrak{m},\mathfrak{n}}(x,t,1)$ by the property $p[v,f]=p[-v,f]$, see Lemma 2.6 (i)) and by taking

$$\begin{aligned}G_i^{\mathfrak{m}}(x,t)&\coloneqq D^\alpha\left(v_i^{\mathfrak{m}}(x)\cdot\nabla(h_i^{\mathfrak{m}}(x,t))^2\right),\\ F_{i,l}^{\mathfrak{m},\mathfrak{n}}(x,t,a)&\coloneqq D^\alpha\left(2v_i^{\mathfrak{m}}(x)\cdot\nabla p[av_l^{\mathfrak{n}},h_{l,t}^{\mathfrak{n}}](x)\right)\end{aligned}$$

for any given multiindex $\alpha=(\alpha_1,\alpha_2)$.

In order to see that the theorem above is a sharpening of Theorem 2.12, recall that the role of the processes a_1^k,a_2^k (given by Theorem 2.12) was (in a sense) to "pick" (among all influences of the set U_i on the set U_j, $i,j\in\{1,2\}$) only the influence of U_1 on U_2 (recall the comments following Theorem 2.12). Here, instead of a pair U_1,U_2 we have to deal with $\mathfrak{M}$ pairs $U_1^{\mathfrak{m}},U_2^{\mathfrak{m}}$ ($\mathfrak{m}=1,\ldots,\mathfrak{M}$) and the role of the processes $a_1^{\mathfrak{m},k},a_2^{\mathfrak{m},k}$ is to "pick" (among all influences of $U_i^{\mathfrak{n}}$ on $U_l^{\mathfrak{m}}$, $i,l\in\{1,2\}$, $\mathfrak{n},\mathfrak{m}\in\{1,\ldots,\mathfrak{M}\}$) only the influence of $U_1^{\mathfrak{m}}$ on $U_2^{\mathfrak{m}}$ for all $\mathfrak{m}\in\{1,\ldots,\mathfrak{M}\}$ (that is for each pair pick only the influence of the first set on the second one). Thus, recalling that the choice of the processes a_1^k,a_2^k (in Section 2.3.3) was based on the "basic processes" b_1,b_2 (recall

(2.83)) having the simple integral property (2.84), we can obtain the processes $a_1^{\mathfrak{m},k}, a_2^{\mathfrak{m},k}$ by finding processes $b_1^{(\mathfrak{m})}, b_2^{(\mathfrak{m})}$, $\mathfrak{m} = 1, \ldots, \mathfrak{M}$ such that an analogous property holds:

$$\int_0^T b_i^{(\mathfrak{n})}(s) f\left(b_l^{(\mathfrak{m})}(s)\right) \mathrm{d}s = \begin{cases} \frac{T}{2}(f(1) - f(0)) & (i,l) = (2,1), \mathfrak{m} = \mathfrak{n}, \\ 0 & \text{otherwise} \end{cases} \tag{3.37}$$

for any $f\colon [-1,1] \to \mathbb{R}$ such that $f(-1) = f(1)$. Such processes can be obtained by letting $b_1^{(1)} := b_1$, $b_2^{(1)} := b_2$ and letting $b_i^{(\mathfrak{m})}$ have 4 times higher frequency than $b_i^{(\mathfrak{m}-1)}$, $i = 1, 2$, $\mathfrak{m} \in \{2, \ldots, \mathfrak{M}\}$, that is

$$b_1^{(\mathfrak{m})}(t) := b_1(4^{\mathfrak{m}-1}t), \qquad b_2^{(\mathfrak{m})}(t) := b_2(4^{\mathfrak{m}-1}t) \tag{3.38}$$

where we extended b_1, b_2 T-periodically to the whole line, see Fig. 3.3. Analogously

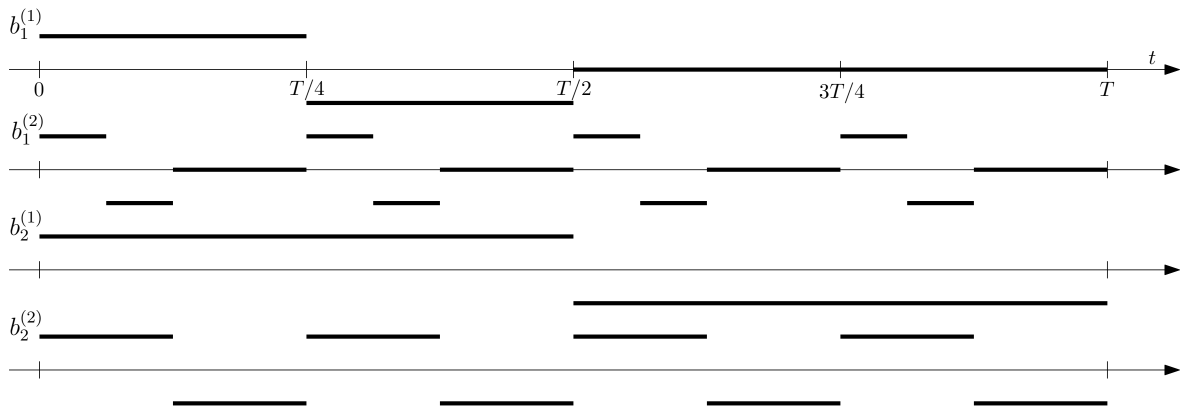

Figure 3.3: The processes $b_1^{(\mathfrak{m})}$, $b_2^{(\mathfrak{m})}$, $\mathfrak{m} = 1, \ldots, \mathfrak{M}$. Here $\mathfrak{M} = 2$.

as in Section 2.3.3 the convergence (3.36) can be obtained by letting, for each k, $b_1^{\mathfrak{m},k}$, $b_2^{\mathfrak{m},k}$ ($\mathfrak{m} = 1, \ldots, \mathfrak{M}$) be oscillations of the above form with frequency increasing with k, that is

$$b_1^{\mathfrak{m},k}(t) := b_1^{(\mathfrak{m})}(kt) = b_1(k4^{\mathfrak{m}-1}t), \quad b_2^{\mathfrak{m},k}(t) := b_2^{(\mathfrak{m})}(kt) = b_2(k4^{\mathfrak{m}-1}t). \tag{3.39}$$

As in Section 2.3.3, the smoothness of the processes can be obtained by smooth approximation of the processes $b_1^{\mathfrak{m},k}, b_2^{\mathfrak{m},k}$, that is by letting $a_1^{\mathfrak{m},k}, a_2^{\mathfrak{m},k} \in C^\infty(\mathbb{R}; [-1,1])$ be such that

$$\left| \left\{ t \in [0,T] \colon a_i^{\mathfrak{m},k}(t) \neq b_i^{\mathfrak{m},k}(t) \right\} \right| \leq \frac{1}{k}, \qquad i = 1, 2, \mathfrak{m} = 1, \ldots, \mathfrak{M}.$$

3.2.3 The new geometric arrangement

In this section we construct the geometric arrangement as described in Section 3.2. That is we need to find $U_1, U_2 \Subset P$ (with disjoint closures) together with the corresponding structures (v_1, f_1, ϕ_1), (v_2, f_2, ϕ_2) and numbers $T > 0$, $\tau \in (0,1)$, $z = (z_1, z_2, 0) \in \mathbb{R}^3$, $X > 0$, $M \in \mathbb{N}$ such that except for (2.50), (2.51) (which was all that we required in the proof of Theorem 2.1, recall Section 2.3) we also have (3.2), (3.3) and (3.12), that is $\{\Gamma_n(G)\}_{n=1,\ldots,M}$ is a family of pairwise disjoint subsets of G_2 (where $G = G_1 \cup G_2 = R(\overline{U_1}) \cup R(\overline{U_2})$),

$$\tau^\xi M \geq 1, \qquad \tau M < 1$$

and

$$f_2^2(y_n) + T v_2(y_n) \cdot F[v_1, f_1](y_n) > \tau^{-2} \left(f_1(R^{-1}x) + f_2(R^{-1}x) \right)^2$$

for all $x \in G$ and $n = 1, \ldots, M$, where $y_n = R^{-1}(\Gamma_n(x))$. The construction builds on the objects defined previously (in Section 2.4) and, remarkably, can be obtained simply by taking $\varepsilon > 0$ smaller, which we present in several steps.

Step 1. Recall some objects from Section 2.4.

Let

$$U, v, f, \phi, F, A, B, C, D, \kappa \quad \text{ and } a', r', s', a'', r'', s'', H, E$$

be as in Section 2.4. In particular, U is a rectangle in P, (v, f, ϕ) is a structure on U, $F = F[v, f]$ is a pressure interaction function corresponding to U, the constants $A, B, C, D \in \mathbb{R}$ are given by the properties of the pressure interaction function F (recall Lemma 2.9), $\kappa = 10^4 C/D$ (recall (2.48)), the numbers $a', r', s', a'', r'', s''$ define the copies $U^{a',r'}$, $U^{a'',r''}$ of U (and the copies of the corresponding structures) in a way that the joint pressure interaction function $H = F + F^{a',r',s'} + F^{a'',r'',s''}$ has certain decay and certain behaviour on the x_1 axis (that is (i)-(iii) from Section 2.4.2 hold), and $E > 0$ is sufficiently small such that the strip $0 < x_2 < E$ is disjoint with $U \cup U^{a',r'} \cup U^{a'',r''}$ and H enjoys certain properties in this strip (that is (iv)-(vi) from Section 2.4.2 hold).

Step 2. Consider disjoint copies of $U \cup U^{a',r'} \cup U^{a'',r''}$ in the x_1 direction.

Let $X > 0$ be sufficiently large so that

$$X > \operatorname{diam}\left(U \cup U^{a',r'} \cup U^{a'',r''} \right), \qquad X > 4|A|,$$
$$2CX^{-4} \sum_{k \in \mathbb{Z}} \left(|k| - \frac{1}{2} \right)^{-4} < 0.01B, \qquad \text{and} \qquad X > 2\kappa E, \tag{3.40}$$

and consider the collection of copies of $U \cup U^{a',r'} \cup U^{a'',r''}$:

$$\left\{ U^{nX,1} \cup U^{a'+nX,r'} \cup U^{a''+nX,r''} \right\}_{n \in \mathbb{Z}} \tag{3.41}$$

together with the structures that are the corresponding translations by $(nX, 0)$ of

$$(v, f, \phi) + \left(v^{a',r',s'}, f^{a',r',s'}, \phi^{a',r'} \right) + \left(v^{a'',r'',s''}, f^{a'',r'',s''}, \phi^{a'',r''} \right),$$

recall (2.104) (see Fig. 3.4). The role of X is to separate these copies (and the corresponding structures) sufficiently far from each other. In particular we see that they have disjoint closures by the first inequality in (3.40). Note also that since each of $U^{nX} \cup U^{a'+nX,r'} \cup U^{a''+nX,r''}$, $n \in \mathbb{Z}$, is a translation in the x_1 direction of $U \cup U^{a',r'} \cup U^{a'',r''}$, it is disjoint with the strip $\{0 < x_2 < E\}$ (recall (iv) in Section 2.4.2), see Fig. 3.4.

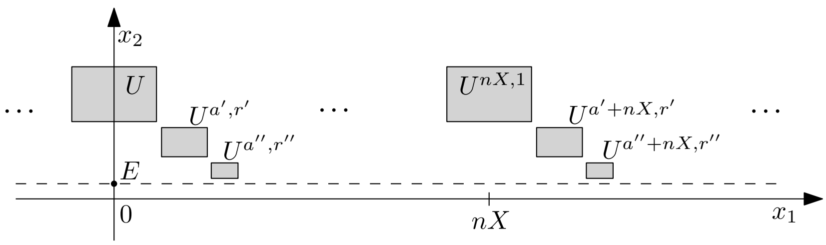

Figure 3.4: The sets $U^{nX,1} \cup U^{a'+nX,r'} \cup U^{a''+nX,r''}$, $n \in \mathbb{Z}$.

Moreover, note that for each $n \in \mathbb{Z}$

$$H(x_1 - nX, x_2) = \left(F^{nX,1} + F^{a'+nX,r',s'} + F^{a''+nX,r'',s''} \right)(x_1, x_2), \qquad (x_1, x_2) \in \mathbb{R}^2,$$

that is $H(x_1 - nX, x_2)$ is the pressure interaction function corresponding to $U^{nX,1} \cup U^{a'+nX,r'} \cup U^{a''+nX,r''}$ (with the structure as pointed out above). We now show that the choice of X above gives that for each $k \in \mathbb{Z}$ the total pressure interaction of the sets (3.41) for $n \neq k$ (and their structures) is very small near $U^{kX} \cup U^{a'+kX,r'} \cup U^{a''+kX,r''}$, which we make precise in the following lemma.

Lemma 3.4. *Given $x_1 \in \mathbb{R}$ let $k \in \mathbb{Z}$ be such that*

$$|x_1 - kX| = \min_{n\in\mathbb{Z}} |x_1 - nX|.$$

Then

$$\sum_{n\neq k} |H(x_1 - nX, x_2)| < 0.01B, \qquad x_2 \in [0, E).$$

Proof. If $n \neq k$ then

$$|x_1 - nX| \geq \left(|n-k| - \frac{1}{2} \right) X,$$

cf. Fig. 3.4. Thus in particular

$$|x_1 - nX| \geq X/2 \geq 2|A|,$$

where we used the fact that $X \geq 4|A|$ (see (3.40)), and we can use the decay of H (see property (iii) of H) to write

$$|H(x_1 - nX, x_2)| \leq 2C|x_1 - nX|^{-4} \leq 2C \left(|n-k| - \frac{1}{2} \right)^{-4} X^{-4}.$$

Summing up in n and using the third inequality in (3.40) we obtain

$$\sum_{n\neq k} |H(x_1 - nX, x_2)| \leq 2CX^{-4} \sum_{n\neq k} \left(|n-k| - \frac{1}{2} \right)^{-4} \leq 0.01B. \qquad \square$$

Thus, for any $M \in \mathbb{N}$ the function

$$H^*(x_1, x_2) \coloneqq \sum_{n=0}^{M-1} H(x_1 - nX, x_2)$$

is the pressure interaction function corresponding to

$$\bigcup_{n=0}^{M-1} \left(U^{nX} \cup U^{a'+nX,r'} \cup U^{a''+nX,r''} \right),$$

and the above lemma and properties (v) and (vi) of H give

(i) $H_1^*(x) \geq -1.02B$ in the strip $\{0 < x_2 < E\}$,

(ii) $H_1^*(x) \geq 6.98B$ for $x \in P$ with $|x_1 - A - (m-1)X| < \kappa E$, $0 < x_2 < E$ for any $m = 1, \dots, M$.

Step 3. Take $\varepsilon > 0$ small, and define v_2, U_2.

Given $\varepsilon > 0$ let $\tau := 0.48\varepsilon$ and

$$r := E/\varepsilon, \qquad d := \kappa r, \qquad M := 1 + \frac{d}{4X}. \tag{3.42}$$

Note each of r, d, M is of order ε^{-1}. Let ε be small such that in addition to (2.107) we also have that M is a positive integer and

$$\tau^{\xi} M \geq 1, \qquad \varepsilon^2 M < \frac{10^{-6} B E^4}{2C}. \tag{3.43}$$

Note that this gives (3.2), which is clear from the first of the two inequalities above and by writing

$$\tau M = \tau + \frac{\tau d}{4X} = \tau + \frac{0.48 \kappa E}{4X} < \tau + \frac{1}{2} < 1,$$

where we used the facts $X > \kappa E/4$ (recall (3.40)) and $\tau < 1/2$ (recall that in fact $\tau < 1/20$ by the first inequality in (2.107)).

Having fixed ε we let (as previously) v_2 be given by Lemma 2.14 and the sets U_2, BOX, $RECT$, $SBOX$ be defined as in (2.110). Note that U_2 encompasses the union of all M copies of $U \cup U^{a',r'} \cup U^{a'',r''}$, that is

$$\bigcup_{n=0}^{M-1} \left(U^{nX,1} \cup U^{a'+nX,r'} \cup U^{a''+nX,r''} \right) \subset (-(d-r), d-r) \times (\varepsilon r, r/10) \tag{3.44}$$

(see Fig. 3.6), which can be verified in the same way as (2.112), except for the use of the inequality $d - r > \max\{1, a' + r', a'' + r''\}$, which can be sharpened using the fifth inequality in (2.107) and the fact that $d > 4r$ (recall (2.106)),

$$\begin{aligned} d - r = \left(\frac{d}{2} - r \right) + \frac{d}{2} &> \frac{d}{4} + \operatorname{diam} \left(U \cup U^{a',r'} \cup U^{a'',r''} \right) \\ &= (M-1)X + \operatorname{diam} \left(U \cup U^{a',r'} \cup U^{a'',r''} \right), \end{aligned}$$

and so (3.44) follows. Let

$$SBOX_m := SBOX + (m-1)(X, 0), \quad m = 1, \ldots, M,$$

and observe that $\{SBOX_m\}_{m=1}^{M}$ is a family of pairwise disjoint subsets of $RECT$ (cf. Fig. 3.5). Indeed, the disjointness follows from the fact that $X > 2\kappa E$ (recall (3.40)), the inclusion $SBOX_1 \subset RECT$ follows as previously (recall the comment following (2.110)) and the inclusion $SBOX_M \subset RECT$ follows by writing

$$(M-1)X + A + \kappa E = \frac{d}{4} + A + \kappa E < \frac{d}{4} + (d-r)/2 < d - r,$$

where we used the second inequality in (2.107) and the fact that $d > 2r$ (recall (2.106)).

Let

$$a := -\kappa r/\varepsilon, \quad \frac{s^2}{r} := 1.04 \left(-\frac{a}{r}\right)^4 B/D$$

(as previously, see (2.114)) and note that then Lemma 2.15 gives

$$1.03B \le F_1^{a,r,s} \le 1.05B \quad \text{and} \quad |F_2^{a,r,s}| \le 0.01\varepsilon B \quad \text{in } BOX. \tag{3.45}$$

Step 4. Define U_1, its structure (v_1, f_1, ϕ_1), and show the lower bound $v_2 \cdot F[v_1, f_1] \ge -1.1\varepsilon B$.

Letting

$$U_1 := \bigcup_{n=0}^{M-1} \left(U^{nX,1} \cup U^{a'+nX,r'} \cup U^{a''+nX,r''} \right) \cup U^{a,r},$$

and

$$\begin{aligned} f_1 &:= \sum_{n=0}^{M-1} \left(f^{nX,1,1} + f^{a'+nX,r',s'} + f^{a''+nX,r'',s''} \right) + f^{a,r,s}, \\ v_1 &:= \sum_{n=0}^{M-1} \left(v^{nX,1,1} + v^{a'+nX,r',s'} + v^{a''+nX,r'',s''} \right) + v^{a,r,s}, \\ \phi_1 &:= \sum_{n=0}^{M-1} \left(\phi^{nX,1} + \phi^{a'+nX,r'} + \phi^{a''+nX,r''} \right) + \phi^{a,r} \end{aligned}$$

we obtain a structure (v_1, f_1, ϕ_1) on U_1. We see that $\overline{U^{a,r}}$ is located to the left of BOX (as previously, see (2.118)) and so, in the view of (3.44),

$$U_1, U_2 \Subset P \ \text{ have disjoint closures.} \tag{3.46}$$

Denoting by F^* the total pressure interaction function,

$$\begin{aligned} F^* := F[v_1, f_1] &= \sum_{n=0}^{M-1} \left(F^{nX,1,1} + F^{a'+nX,r',s'} + F^{a''+nX,r'',s''} \right) + F^{a,r,s} \\ &= H^* + F^{a,r,s}, \end{aligned}$$

we see that properties (i), (ii) of H^* above (see Step 2) and (3.45) give

$$\begin{cases} F_1^* \ge 0.01B & \text{in } (-(d-r), d-r) \times (0, \varepsilon r) \supset RECT, \\ F_1^* \ge 8B & \text{in } SBOX_m, \quad m = 1, \ldots, M, \end{cases} \tag{3.47}$$

cf. (2.115). Moreover

$$v_2 \cdot F^* \geq -1.1\varepsilon B \qquad \text{in } BOX, \tag{3.48}$$

which is an analogue of the previous relation (2.116) and which we now verify. Let $x \in \operatorname{supp} v_2$ (otherwise the claim is trivial).

Case 1. $x \in [0,(M-1)X] \times \{0\} + B(0,r/10)$. In this case $x_2 \in (0,\varepsilon r)$ (see Fig. 3.5) and

$$-(d-r) < r/10 < x_1 < (M-1)X + r/10 = d/4 + r/10 < d-r,$$

where the left-most and the right-most inequalities follow from the fact that $d > 10^4 r$ (recall (2.106)). Thus $x \in (-(d-r), d-r) \times (0,\varepsilon r)$ (see Fig. 3.5) and consequently

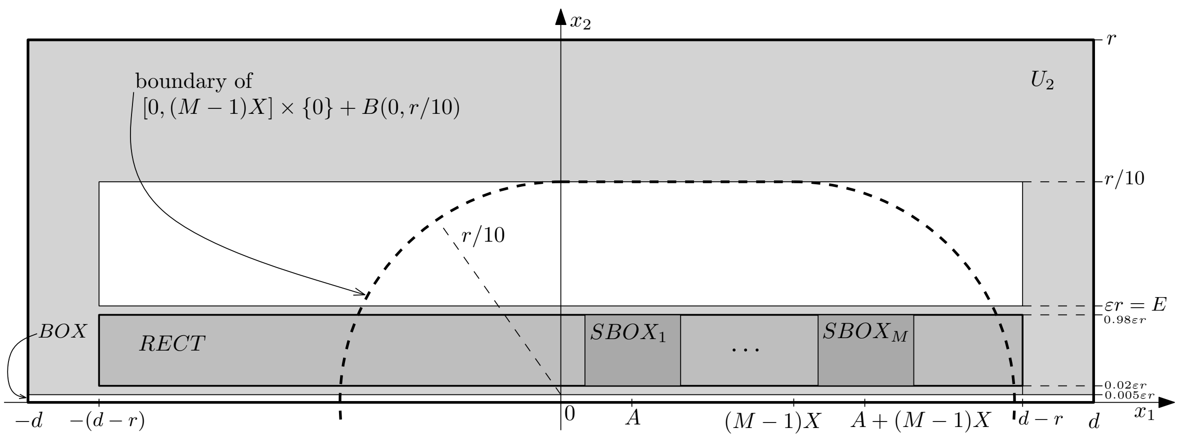

Figure 3.5: The sets U_2, BOX, $RECT$, and $SBOX_m$, $m = 1,\dots,M$ (compare with Fig. 2.13). Note that some proportions are not conserved on this sketch.

the choice of v_2 (see Lemma 2.14 (iii)) and (3.47) give

$$v_2(x) \cdot F^*(x) = v_{21}(x) F_1^*(x) \geq 0.01 B v_{21}(x) > 0 > -1.1\varepsilon B.$$

Case 2. $x \notin [0,(M-1)X] \times \{0\} + B(0,r/10)$. In this case

$$|H^*(x)| \leq 0.01\varepsilon^2 B, \tag{3.49}$$

which is an analogue of (2.117) and which follows from the decay of H (that is property (iii) in Section 2.4.2). Indeed, since in this case

$$|(x_1 - (n-1)X, x_2)| \geq r/10, \qquad n = 1,\dots,M,$$

and since $r > 20|A|$ (recall (2.107)) we obtain

$$|(x_1 - (n-1)X, x_2)| \geq 2|A| \qquad n = 1, \ldots, M.$$

Thus

$$\begin{aligned}|H^*(x_1,x_2)| &\leq \sum_{n=0}^{M-1} |H(x_1 - nX, x_2)| \leq 2C \sum_{n=1}^{M} |(x_1 - nX, x_2)|^{-4} \\ &\leq 2C \sum_{n=1}^{M} \left(\frac{10}{r}\right)^4 = 2 \cdot 10^4 CM\varepsilon^4 E^{-4} < 0.01\varepsilon^2 B,\end{aligned}$$

where we used (3.43) in the last step. Hence we obtained (3.49), and so, using the properties of the choice of v_2 (that is Lemma 2.14 (iii)) and the bounds on $F^{a,r,s}$ (see (3.45)) we obtain (3.48) by writing

$$\begin{aligned}v_2(x) \cdot F^*(x) &= v_2(x) \cdot H(x) + v_{21}(x) F_1^*(x) + v_{22}(x) F_2^*(x) \\ &\geq -2(0.01\varepsilon^2 B) - \varepsilon^2(1.05B) - \frac{\varepsilon}{2}(0.01B\varepsilon) \\ &= -\varepsilon^2 B(0.02 + 1.05 + 0.005) \geq -1.1\varepsilon^2 B.\end{aligned}$$

Step 5. Verify (3.3).

As previously let $z := (A, \varepsilon r/2, 0)$ and observe that

$$R^{-1}(\Gamma_m(R(BOX))) \subset SBOX_m \qquad m = 1, \ldots, M, \tag{3.50}$$

which follows in the same way as the prevous property (2.111). In fact (2.111) corresponds to the case $m = 1$, and the claim for other values of m follows by translating in the x_1 direction both sides of (2.111) by multiplies of X, see Fig. 3.6. Thus, since the sets $SBOX_m$, $m = 1, \ldots, M$, are pairwise disjoint,

$$\{R^{-1}(\Gamma_m(R(BOX)))\}_{m=1}^M \text{ is a family of disjoint sets.}$$

We now show that

$$\begin{aligned}\left\{R^{-1}\left(\Gamma_m\left(\overline{U^{a,r}}\right)\right)\right\}_{m=1}^M \text{ is a pairwise disjoint family of subsets of } RECT \\ \text{which are located to the left of } SBOX_1,\end{aligned} \tag{3.51}$$

which is an analogue of the previous relation (2.119), see Fig. 3.6. Here "to the left of" refers to the property that the x_1 coordinate of any point of $R^{-1}\left(\Gamma_M\left(\overline{U^{a,r}}\right)\right)$ is strictly

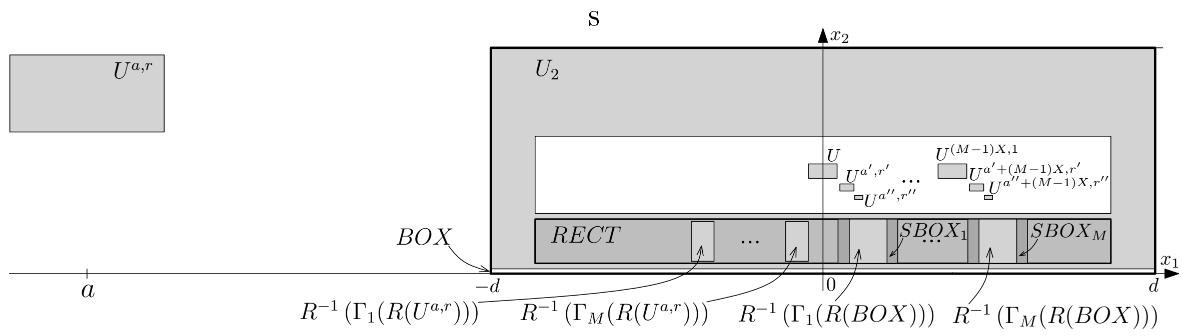

Figure 3.6: The geometric arrangement for Theorem 2.2 (compare with Fig. 2.14). Note that proportions are not conserved on this sketch.

less than the x_1 coordinate of any point of $SBOX_1$; since both $R^{-1}\left(\Gamma_M\left(\overline{U^{a,r}}\right)\right)$ and $SBOX_1$ are rectangles, this is simply

$$\tau(a+r)+A+(M-1)X<A-\kappa E.$$

This inequality can be verified using the facts $\varepsilon<1/10$ (recall (2.107)) and $\kappa>1$ (recall (2.106)) by writing

$$\begin{aligned}\tau(a+r)+(M-1)X &= \tau r(1-\kappa/\varepsilon)+d/4=0.48r(\varepsilon-\kappa)+\kappa r/4\\ &=0.48r\varepsilon-0.23\kappa r<0.48r\varepsilon-2\varepsilon\kappa r=\varepsilon r(0.48-2\kappa)\\ &<-\kappa\varepsilon r=-\kappa E,\end{aligned}$$

as required. Property (3.51) is now clear by recalling that the previous property (2.119) gives

$$R^{-1}\left(\Gamma_1\left(\overline{U^{a,r}}\right)\right)\subset RECT,$$

and that the fact $X>2E$ (recall (3.40)) gives $X>2\tau r$, which shows that the sets $R^{-1}\left(\Gamma_m\left(\overline{U^{a,r}}\right)\right)$ are pairwise disjoint (recall each of these sets is a rectangle whose length (in the x_1 direction) is $2\tau r$, cf. the comment following (2.119)).

Properites (3.50) and (3.51) give that

$$\left\{R^{-1}\left(\Gamma_m(G)\right)\right\}_{m=1}^{M}\quad\text{is a family of disjoint subsets of } RECT$$

(recall $G=R(\overline{U_1}\cup\overline{U_2})$), which gives (3.3). Indeed,

$$\Gamma_m(G)\subset R(RECT)\subset R(\overline{U_2})=G_2,\quad m=1,\dots,M,$$

and the disjointness follows from the disjointness of the cylindrical projections.

Step 6. Define T, f_2 and ϕ_2 and show the remaining claims (3.11) and (3.12).

Let T, f_2, ϕ_2 be defined as previously (see Section 2.4.5). Then (v_2, f_2, ϕ_2) is a structure on U_2 and (3.11) and (3.12) follow in the same way as (2.50), (2.51) in Section 2.4.5 by making the following replacements. Replace y by y_n and $SBOX$ by $SBOX_n$, $n = 1, \ldots, M$, and use the relations (3.48), (3.50), (3.47), (3.51) instead of the previous relations (2.116), (2.111), (2.115), (2.119) (respectively).

3.3 Proof of Theorems 2.3 and 2.4

In this section we construct vector fields which, except for the Navier–Stokes inequality (1.24),

$$u \cdot (\partial_t u - \nu \Delta u + (u \cdot \nabla) u + \nabla p) \leq 0$$

(with some ν), also satisfy the approximate equality (2.5),

$$\| u \cdot (\partial_t u - \nu \Delta u + (u \cdot \nabla) u + \nabla p) \|_{L^\infty} \leq \vartheta,$$

or equivalently

$$\| \partial_t |u|^2 - 2\nu \, u \cdot \Delta u + u \cdot \nabla (|u|^2 + 2p) \|_{L^\infty} \leq 2\vartheta,$$

In the case of Theorem 2.3 we focus on the case $\nu = 0$ (in which the inequality (1.24) should perhaps be called the *Euler inequality*) and we construct a vector field which blows-up, wheras in the case of Theorem 2.4 we construct a smooth vector field satisfying the above inequalities for all $\nu \in [0,1]$, but which does not blow up. Instead it admits a norm inflation effect over the time interval $[0,1]$.

3.3.1 Theorem 2.3

In this section we prove Theorem 2.3, that is given $\xi \in (0,1)$, $\vartheta > 0$ we construct a vector field $\mathfrak{u}$ satisfying conditions (i)-(iv) of Theorem 2.2 with $\nu_0 = 0$ together with an additional property

$$\| \partial_t |u|^2 + u \cdot \nabla (|u|^2 + 2p) \|_{L^\infty} \leq 2\vartheta.$$

We explain below that $\mathfrak{u}$ can be obtained by replacing δ in (3.16) by

$$\delta_j := \min \left\{ \delta, 2\tau^{4j} \vartheta / 3 \right\}, \tag{3.52}$$

and by using the construction from Section 3.2. Note that such a trick immediately gives $\nu_0 = 0$ since the replacement of δ by δ_j in (2.68) gives

$$\nu_0 \sup_{x\in R(U_1\setminus \operatorname{supp}\phi_1)} |u[0,f_1](x)\cdot \Delta u[0,f_1](x)| \le \tau^{4j}\vartheta/6,$$

and so taking $j\to\infty$ implies $\nu_0=0$.

We now make the construction precise.

Step 1. Construct the geometric arrangement as in Section 3.2.3.

Step 2. Let $j\geq 0$.

Step 3. Let h_1, h_2 be as (3.16) but with $\delta>0$ replaced by δ_j that is

$$\begin{aligned} h_{1,t}^2 &:= f_1^2 - 2t\delta_j\phi_1, \\ h_{2,t}^2 &:= f_2^2 - 2t\delta_j\phi_2 + \int_0^t v_2\cdot F[v_1,h_{1,r}]\,\mathrm{d}r, \end{aligned} \tag{3.53}$$

and let $h_t := h_{1,t}+h_{2,t}$. Note that, as in Section 3.2, $h_1,h_2 \in C^\infty(P\times(-\delta_j,T+\delta_j);[0,\infty))$,

$$(v_i,h_{i,t},\phi_i) \text{ is a structure on } U_i \qquad \text{for } t\in(-\delta_j,T+\delta_j), i=1,2,$$

and

$$h_{2,T}^2(y_n) > \tau^{-2}\left(f_1(R^{-1}x)+f_2(R^{-1}x)\right)^2+\theta \tag{3.54}$$

for $x\in G$, $n=1,\ldots,M$, where $y_n = R^{-1}(\Gamma_n(x))$.

Step 4. Let

$$h_t^{(j)}(x_1,x_2) := \sum_{m\in M(j)} h_t(\pi_m^{-1}(\tau^j x_1),x_2),$$

and let $\varrho_j>0$ and $v^{(j)}\in C^\infty\left(\mathbb{R}^3\times(-\varrho_j,T+\varrho_j);\mathbb{R}^3\right)$ be such that conditions (i)-(iv) of Proposition 3.2 are satisfied with $\nu_0=0$ and

$$\partial_t\left|v^{(j)}\right|^2 + v^{(j)}\cdot\nabla\left(\left|v^{(j)}\right|^2+2\overline{p}^{(j)}\right) \geq -2\tau^{4j}\vartheta \tag{3.55}$$

in $\mathbb{R}^3\times[0,T]$ (where $\overline{p}^{(j)}$ is the pressure function corresponding to $v^{(j)}$).

To this end, we repeat the proof of Proposition 3.2 with δ replaced by δ_j and, in order to obtain the extra property (3.55), we modify the calculations from "Case 1" and "Case 2" from Section 3.2.1 as follows. Fix $x\in P$, $t\in[0,T]$ and write, for brevity,

$v = v^{(j)}$, $\overline{p} = \overline{p}^{(j)}$.

Case 1. $\phi_1^{\mathfrak{m}}(x) + \phi_2^{\mathfrak{m}}(x) < 1$ for all $\mathfrak{m} \in \{1, \ldots, \mathfrak{M}\}$. Then

$$\begin{aligned}\partial_t |v(x,0,t)|^2 &= -2\delta_j \sum_{\mathfrak{m}=1}^{\mathfrak{M}} (\phi_1^{\mathfrak{m}}(x) + \phi_2^{\mathfrak{m}}(x)) \\ &\geq -2\tau^{4j}\vartheta \\ &= -2\tau^{4j}\vartheta - v(x,0,t)\cdot \nabla \left(|v(x,0,t)|^2 + 2\overline{p}(x,0,t)\right).\end{aligned}$$

Case 2. $\phi_1^{\mathfrak{m}}(x) + \phi_2^{\mathfrak{m}}(x) = 1$ for some $\mathfrak{m} \in \{1, \ldots, \mathfrak{M}\}$. In this case (3.32) gives

$$\begin{aligned}|v_i^{\mathfrak{m}}| &\left(\left| \nabla (q_{i,t}^{\mathfrak{m},k})^2 - \nabla (h_{i,t}^{\mathfrak{m}})^2 \right| + 2 \sum_{\mathfrak{n}=1}^{\mathfrak{M}} \sum_{j=1,2} \left| \nabla p[a_j^{\mathfrak{n},k}(t) v_j^{\mathfrak{n}}, q_{j,t}^{\mathfrak{n},k}] - \nabla p[a_j^{\mathfrak{n},k}(t) v_j^{\mathfrak{n}}, h_{j,t}^{\mathfrak{n}}] \right| \right) \\ &\leq \delta_j/2\end{aligned}$$

in P ($t \in [0,T]$, $i = 1,2$), and so

$$\begin{aligned}\partial_t |v(x,0,t)|^2 &= \partial_t q_{1,t}^{\mathfrak{m},k}(x)^2 + \partial_t q_{2,t}^{\mathfrak{m},k}(x)^2 \\ &= -2\delta_j - \left(a_1^{\mathfrak{m},k}(t) v_1^{\mathfrak{m}}(x) + a_2^{\mathfrak{m},k}(t) v_2^{\mathfrak{m}}(x) \right) \cdot \nabla \Bigg(\left(h_{1,t}^{\mathfrak{m}}(x)\right)^2 + \left(h_{2,t}^{\mathfrak{m}}(x)\right)^2 \\ &\qquad + 2 \sum_{\mathfrak{n}=1}^{\mathfrak{M}} \left(p[a_1^{\mathfrak{n},k}(t) v_1^{\mathfrak{n}}, h_{1,t}^{\mathfrak{n}}](x) + p[a_2^{\mathfrak{n},k}(t) v_2^{\mathfrak{n}}, h_{2,t}^{\mathfrak{n}}](x) \right) \Bigg) \\ &\geq -3\delta_j - \left(a_1^{\mathfrak{m},k}(t) v_1^{\mathfrak{m}}(x) + a_2^{\mathfrak{m},k}(t) v_2^{\mathfrak{m}}(x) \right) \cdot \nabla \Bigg(q_{1,t}^{\mathfrak{m},k}(x)^2 + q_{2,t}^{\mathfrak{m},k}(x)^2 \\ &\qquad + 2 \sum_{\mathfrak{n}=1}^{\mathfrak{M}} \left(p[a_1^{\mathfrak{n},k}(t) v_1^{\mathfrak{n}}, q_{1,t}^{\mathfrak{n},k}](x) + p[a_2^{\mathfrak{n},k}(t) v_2^{\mathfrak{n}}, q_{2,t}^{\mathfrak{n},k}](x) \right) \Bigg) \\ &= -3\delta_j - v_1(x,0,t)\partial_{x_1} \left(|v(x,0,t)|^2 + 2\overline{p}(x,0,t) \right) \\ &\qquad - v_2(x,0,t)\partial_{x_2} \left(|v(x,0,t)|^2 + 2\overline{p}(x,0,t) \right) \\ &\geq -2\tau^{4j}\vartheta - v(x,0,t)\cdot \nabla \left(|v(x,0,t)|^2 + 2\overline{p}(x,0,t)\right),\end{aligned}$$

since (as in Section 3.2.1) $\partial_{x_3} |v(x,0,t)|^2 = \partial_{x_3} \overline{p}(x,0,t) = 0$.

Step 5. Let $\mathfrak{u}$ be as in (3.24), that is

$$\mathfrak{u}(t) := \begin{cases} u^{(j)}(t) & \text{if } t \in [t_j, t_{j+1}) \text{ for some } j \geq 0, \\ 0 & \text{if } t \geq T_0, \end{cases}$$

where

$$u^{(j)}(x_1, x_2, x_3, t) := \tau^{-j} v^{(j)}(\tau^{-j} x_1, \gamma^{-j}(x_2), \tau^{-j} x_3, \tau^{-2j}(t - t_j)).$$

Then (3.55) gives

$$\partial_t \left|u^{(j)}\right|^2 + u^{(j)} \cdot \nabla \left(\left|u^{(j)}\right|^2 + 2p \right) \geq -2\vartheta, \qquad j \geq 0,$$

and the rest of the claims of Theorem 2.3 follow as in Section 3.2.

3.3.2 Theorem 2.4

We will construct $T > 0$, $\nu_0 > 0$, $\eta > 0$ and a divergence-free vector field $u \in C^\infty(\mathbb{R}^3 \times (-\eta, T+\eta); \mathbb{R}^3)$ such that $\operatorname{supp} u(t) = G$ for all t (where $G \subset \mathbb{R}^3$ is compact),

$$\|u(T)\|_{L^\infty} \geq \mathcal{N} \|u(0)\|_{L^\infty} \tag{3.56}$$

and

$$-\frac{2\vartheta}{T^2 \nu_0} \leq \partial_t |u|^2 - 2\nu\, u \cdot \Delta u + u \cdot \nabla(|u|^2 + 2p) \leq 0 \qquad \text{in } \mathbb{R}^3 \times (-\eta, T+\eta) \tag{3.57}$$

for any $\nu \in [0, \nu_0]$, where p is the pressure function corresponding to u. A solution of the theorem (which corresponds to the case $T = \nu_0 = 1$) can be obtained by a simple rescaling; namely

$$\mathfrak{u}(x,t) := \sqrt{\nu_0 T} u \left(\sqrt{T/\nu_0} x, Tt \right)$$

is then a solution to Theorem 2.4.

Let $T > 0$, $\nu_0 \in (0,1)$, $\eta > 0$ and u be as in the proof of Theorem 2.1 (that is recall Proposition 2.11, (2.68), (2.121)). We now verify that such choice satisfies the required properties given we take ε (the "sharpness" of the geometric arrangement, recall (2.107)) and δ (a part of the definition of h, recall Lemma 2.10) sufficiently small to account for (3.56), (3.57). To this end, observe that the required smoothness, the divergence-free property, the condition $\operatorname{supp} u(t) = G$ and the right-most inequality in (3.57) follow directly from Proposition 2.11. Moreover observe that

$$\|u(0)\|_\infty = \|h_0\|_\infty = \|f_1 + f_2\|_\infty,$$

whereas, from (2.64),

$$\|u(T)\|_\infty^2 \geq \|h_T\|_\infty^2 - \theta \geq \tau^{-2} \|f_1 + f_2\|_\infty^2.$$

Thus

$$\|u(T)\|_\infty \geq \mathcal{N} \|u(0)\|_\infty$$

given $\tau^{-1} \geq \mathcal{N}$, that is provided $\varepsilon > 0$ is small such that $\varepsilon \leq (0.48\mathcal{N})^{-1}$, in addition to the smallness requirements of the geometric arrangement (2.107). Note that making the value of ε smaller we also make T larger.

In order to obtain the left-most inequality in (3.57) we perform similar calculation as in Step 4 in the previous section given $\delta > 0$ is small as in Lemma 2.10 and additionally

$$\delta < \vartheta/2T^2.$$

Indeed, since (2.80) gives

$$2\nu_0 \left| u(x,0,t) \cdot \Delta u(x,0,t) \right| \leq \delta$$

and since $\nu_0 < 1$ we write in the case $\phi_1(x)+\phi_2(x) < 1$ (that is Case 1 in Section 2.3.2)

$$\begin{aligned}
\partial_t |u(x,0,t)|^2 =& \partial_t q_{1,t}^k(x)^2 + \partial_t q_{2,t}^k(x)^2 \\
=& -2\delta(\phi_1(x)+\phi_2(x)) \\
>& -2\delta \\
\geq& -2\vartheta/T^2\nu_0 - u(x,0,t)\cdot\nabla\left(|u(x,0,t)|^2 + 2p(x,0,t)\right) \\
&+ 2\nu\, u(x,0,t)\cdot\Delta u(x,0,t)
\end{aligned}$$

for all $\nu \in [0,\nu_0]$, $t \in [0,T]$, where we also used (2.44). In the case $\phi_1(x)+\phi_2(x) = 1$ (that is Case 2 in Section 2.3.2) we use (2.79) to obtain

$$\begin{aligned}
\partial_t |u(x,0,t)|^2 =& \partial_t q_{1,t}^k(x)^2 + \partial_t q_{2,t}^k(x)^2 \\
=& -2\delta - \left(a_1^k(t)v_1(x) + a_2^k(t)v_2(x)\right)\cdot\nabla\left(h_{1,t}(x)^2 + h_{2,t}(x)^2\right. \\
&\left. + 2p[a_1^k(t)v_1, h_{1,t}](x) + 2p[a_2^k(t)v_2, h_{2,t}](x)\right) \\
\geq& -3\delta - \left(a_1^k(t)v_1(x) + a_2^k(t)v_2(x)\right)\cdot\nabla\left(q_{1,t}^k(x)^2 + q_{2,t}^k(x)^2\right. \\
&\left. + 2p[a_1^k(t)v_1, q_{1,t}^k](x) + 2p[a_2^k(t)v_2, q_{2,t}^k](x)\right) \\
=& -3\delta - u(x,0,t)\cdot\nabla\left(|u(x,0,t)|^2 + 2p(x,0,t)\right) \\
\geq& -2\vartheta/T^2\nu_0 - u(x,0,t)\cdot\nabla\left(|u(x,0,t)|^2 + 2p(x,0,t)\right) \\
&+ 2\nu\, u(x,0,t)\cdot\Delta u(x,0,t)
\end{aligned}$$

for all $\nu \in [0,\nu_0]$, $t \in [0,T]$, where we also used (2.32) and (2.37) in the fourth step.

Chapter 4

The surface growth model

Here we will consider the scalar model of surface growth

$$u_t + u_{xxxx} + \partial_{xx} u_x^2 = 0 \tag{4.1}$$

on the one-dimensional torus $\mathbb{T}$, under the assumption that $\int_{\mathbb{T}} u = 0$; we refer to this in what follows as the surface growth model.

As pointed out in the introduction, Blömker & Romito (2009, 2012) observed that this model shares many striking similarities with the three-dimensional Navier–Stokes equations. In particular, in their 2009 paper Blömker & Romito proved local existence in the critical space $\dot{H}^{1/2}$ and (spatial) smoothness for solutions bounded in $L^{8/(2\alpha-1)}((0,T); H^\alpha)$ for any $1/2 < \alpha < 9/2$; in this 2012 paper they prove local existence in a critical space of a similar type to that occurring in the paper by Koch & Tataru (2001) for the Navier–Stokes equations.

The aim of this chapter is to prove partial regularity results for (4.1) that are analogues of those proved by Caffarelli, Kohn, & Nirenberg (1982) for the Navier–Stokes equations. Perhaps surprisingly their inductive method does not seem well adapted to (4.1), and instead we will use the rescaling approach of Lin (1998) and Ladyzhenskaya & Seregin (1999). The main issue in following the approach of Caffarelli, Kohn, & Nirenberg is that the biharmonic heat kernel, given in the one-dimensional case by

$$K(x,t) = \alpha t^{-1/4} f(|x| t^{-1/4}), \qquad \text{where} \qquad f(x) = \int_0^\infty \mathrm{e}^{-s^4} \cos(xs)\,\mathrm{d}s$$

and α is a normalising constant (see Ferrero et al. (2008)), takes negative values so cannot be used as the basis of the construction of a suitable sequence of test functions for use in the local energy inequality.

We seek to bound the dimension of the space-time singular set, which we take here to be defined by (1.29), that is

$$S = \{(x,t) \in \mathbb{T} \times [0,\infty) : \ u \text{ is not space-time Hölder continuous} \\ \text{in any neighbourhood of } (x,t)\}.$$

Note that if u is spatially Hölder continuous on $\mathbb{T}$ with some exponent $\theta \in (0,1)$ then $u \in H^\alpha(\mathbb{T})$ for all $0 < \alpha < \theta$, using the Sobolev–Slobodeckii characterisation of $H^\alpha(\mathbb{T})$ as the collection of all functions such that

$$\int_{\mathbb{T}} \int_{\mathbb{T}} \frac{|f(x)-f(y)|^2}{|x-y|^{1+2\alpha}} \, \mathrm{d}x \, \mathrm{d}y < \infty$$

(see Di Nezza, Palatucci, & Valdinoci, 2012); it follows (using arguments from Blömker & Romito (2009)) that if u is space-time Hölder continuous on $[0,T] \times \mathbb{T}$ then u is spatially smooth. However, the condition $u \in L^\infty_t L^\infty_x$ (which is the surface growth model equivalent of the $L^\infty_t L^3_x$ regularity for the Navier–Stokes equations, see Escauriaza, Seregin, & Sv̌erák, 2003) is not yet known to be sufficient for the regularity of the surface growth model. This is why we do not use local essential boundedness in our definition of S.

In fact, it is not known whether local essential boundedness implies (local) Hölder continuity. Neither, it is not clear that local Hölder continuity on a space-time domain implies smoothness on such a domain. It is therefore not entirely clear whether or not the definition of the singular set in (1.29) is the correct one for the surface growth model. However, the result which we discuss in Chapter 5, suggests (in a sense) that such a definition is optimal.

The results of this chapter have been posted on the ArXiv (see Ożański & Robinson (2017)) and have been submitted for publication.

The structure of this chapter is as follows. In the remainder of this section we introduce some notation, in Section 4.2 we introduce the notion of suitable weak solutions and we show global-in-time existence of such solutions for any initial condition $u_0 \in L^2$ with zero mean. In Section 4.3 we introduce a "nonlinear parabolic Poincaré inequality", which is vital for both of our partial regularity results and a concept of independent interest. We then prove two local regularity results for the surface growth model, the first in terms of u_x (Section 4.4) and the second one in terms of u_{xx} (Section 4.5). As a consequence we can show that the (upper) box-counting dimension of the space-time singular set is no larger than 7/6, and that its one-dimensional parabolic Hausdorff measure is zero.

4.1 Notation

With $z = (x, t)$ we define the centred[1] parabolic cylinder $Q(z, r)$ to be

$$Q(z, r) = (x - r, x + r) \times (t - r^4, t + r^4).$$

Note that the 'cylinder' here is in fact a rectangle. We often use the notation Q_r for a cylinder $Q(z, r)$ for some z. Set

$$f_{z,r} := \fint_{Q(z,r)} f = \frac{1}{|Q(z,r)|} \int_{Q(z,r)} f. \tag{4.2}$$

We set $L^2 = L^2(\mathbb{T})$, $H^k = H^k(\mathbb{T})$, and $W^{k,p} = W^{k,p}(\mathbb{T})$ ($k \geq 0$, $p \geq 1$), function spaces consisting of periodic functions: for example $W^{k,p}$ is the completion of the space of smooth and periodic functions on $\mathbb{T}$ in the $W^{k,p}$ norm. The norm on H^k is equivalent to

$$\left(\sum_{n \in \mathbb{Z}} \left(1 + n^{2k}\right) |\hat{f}(n)|^2 \right)^{1/2},$$

where $\hat{f}(n)$ denotes the n-th Fourier coefficient of f. We write $\| \cdot \|$ to denote the L^2 norm and we write a dot "$\cdot$" above a function space to denote the closed subspace of functions with zero integral so that, for example,

$$\dot{H}^k := \left\{ f \in H^k \, : \, \int_{\mathbb{T}} f = 0 \right\}, \quad k \geq 0.$$

We will also write

$$\|f\|_{\dot{H}^k} = \left(\sum_{n \in \mathbb{Z}} n^{2k} |\hat{f}(n)|^2 \right)^{1/2} = c \|\partial_x^k f\|$$

to denote the H^k seminorm. Note that if $u \in \dot{H}^k$ then

$$\|u\|_{\dot{H}^s} \leq \|u\|_{\dot{H}^k} \qquad \text{for all } s \leq k$$

and hence that $\|u\|_{H^s} \leq c_k \|\partial_x^k u\|$. We will use the Sobolev interpolation,

$$\|u\|_{H^s} \leq \|u\|_{H^{s_1}}^{\theta} \|u\|_{H^{s_2}}^{1-\theta} \tag{4.3}$$

and a similar inequality for the seminorms, where $s_1 \leq s \leq s_2$ and $s = \theta s_1 + (1 - \theta) s_2$.

We write $\int := \int_{\mathbb{T}}$ and, given $T > 0$, we denote the space of smooth functions that are periodic with respect to the spatial variable and compactly supported in a time interval I by $C_0^\infty(\mathbb{T} \times I)$. We denote any universal constant by a C or c.

[1] Note that in many papers $Q(z, r)$ is used for the 'non-anticipating' cylinder which in this case would be $B_r(x) \times (t - r^4, t]$.

4.2 Suitable weak solutions

We first recall Definition 1.8 of a weak solution of the problem (4.1).

Definition 4.1 (Weak solution). *We say that u is a (global-in-time)* weak solution *of the surface growth initial value problem*

$$\begin{cases} u_t = -u_{xxxx} - \partial_{xx} u_x^2, \\ u(0) = u_0 \in \dot{L}^2, \end{cases} \tag{4.4}$$

if for every $T > 0$

$$u \in L^\infty((0,T); \dot{L}^2) \cap L^2((0,T); \dot{H}^2) \tag{4.5}$$

and

$$-\int_0^T \int \left(u\,\phi_t - u_{xx}\phi_{xx} - u_x^2 \phi_{xx} \right) = \int u_0 \phi(0) \tag{4.6}$$

for all $\phi \in C_0^\infty(\mathbb{T} \times [0,T))$.

Note that a simple procedure of cutting off ϕ in time (and an application of the Lebesgue Differentiation Theorem) gives that (4.6) is equivalent to

$$\int u(t)\phi(t) - \int_s^t \int \left(u\,\phi_t - u_{xx}\phi_{xx} - u_x^2 \phi_{xx} \right) = \int u(s)\phi(s) \tag{4.7}$$

being satisfied for all $\phi \in C_0^\infty(\mathbb{T} \times [0,T))$ and almost all s,t with $0 \le s < t$ (including $s = 0$, in which case $u(0) = u_0$).

Note also that it follows from the regularity (4.5) enjoyed by any weak solution that

$$u_x \in L^{10/3}((0,T); L^{10/3}). \tag{4.8}$$

Indeed, using Sobolev interpolation (4.3), for any $0 \le s \le 2$ we have

$$\|u\|_{H^s} \le \|u\|_{L^2}^{1-s/2} \|u\|_{H^2}^{s/2},$$

and so the 1D embedding $H^s \subset L^p$ when $s = 1/2 - 1/p$ gives

$$\|u_x\|_{L^p} \le \|u\|_{L^2}^{(2+p)/4p} \|u\|_{H^2}^{(3p-2)/4p},$$

and so $u_x \in L^{8p/(3p-2)}((0,T); L^p)$; in particular $u_x \in L^{10/3}((0,T); L^{10/3})$.

We now briefly recall the proof of the existence of global-in-time weak solutions to the surface growth initial value problem for any initial data $u_0 \in \dot{L}^2$. We give a sketch of the proof (due to Stein & Winkler, 2005) since it will be required in showing the local energy inequality (Theorem 4.4).

Theorem 4.2 (Existence of weak solutions). *For each $u_0 \in \dot{L}^2$ there exists at least one weak solution of the surface growth initial value problem* (4.4).

Proof (sketch). Fix $T > 0$ and take $N \in \mathbb{N}$, let $\tau := T/N$ denote the time step, set $u_0^\tau := u_0$ and, for $k = 1, \ldots, N$, let $u_k^\tau \in \dot{H}^2$ be a solution of the implicit Euler scheme

$$\int \frac{u_k^\tau - u_{k-1}^\tau}{\tau} \psi = -\int \partial_{xx} u_k^\tau \psi_{xx} - \int \left(\partial_x u_k^\tau\right)^2 \psi_{xx} \tag{4.9}$$

for all $\psi \in \dot{H}^2$. The existence of such u_k^τ can be shown using the Lax–Milgram Lemma and the Leray–Schauder fixed point theorem.

For $t \in [(k-1)\tau, k\tau)$, $k \in \{1, \ldots, N\}$, let

$$\begin{cases} u^\tau(x,t) := \frac{k\tau - t}{\tau} u_{k-1}^\tau(x) + \frac{t-(k-1)\tau}{\tau} u_k^\tau(x), \\ \overline{u}^\tau(x,t) := u_k^\tau(x). \end{cases}$$

In other words u^τ denotes the linear approximation between the neighbouring u_k^τ's, and $\overline{u}^\tau$ denotes the next u_k^τ.

Letting $\phi := u_k^\tau$ in (4.9) and observing the cancellation

$$\int \left(\partial_x u_k^\tau\right)^2 \partial_{xx} u_k^\tau = 0$$

we obtain

$$\int \left(u_k^\tau\right)^2 + \tau \int \left(\partial_{xx} u_k^\tau\right)^2 = \int \left(u_{k-1}^\tau\right)^2, \qquad k \geq 1,$$

from which, by summing in k, we obtain the energy inequality for $\overline{u}^\tau$,

$$\|\overline{u}^\tau(t)\|^2 + \int_0^t \|\partial_{xx} \overline{u}^\tau(s)\|^2 \mathrm{d}s \leq \|u_0\|^2, \qquad t \in (0,T), \tag{4.10}$$

and similarly for u^τ,

$$\|u^\tau(t)\|^2 + \int_0^t \|\partial_{xx} u^\tau(s)\|^2 \mathrm{d}s \leq C\|u_0\|^2, \qquad t \in (0,T). \tag{4.11}$$

Furthermore, observe that for every $\psi \in \dot{H}^2$ and every $t \in [0,T)$ we have

$$\int \partial_t u^\tau(t) \psi = \int \frac{u_k^\tau - u_{k-1}^\tau}{\tau} \psi = -\int \left(\partial_{xx} \overline{u}^\tau(t) + (\partial_x \overline{u}^\tau(t))^2\right) \psi_{xx},$$

where $k \geq 1$ is such that $t \in [(k-1)\tau, k\tau)$. Thus, since each u_k^τ, $k \geq 0$, has zero mean, the above equality holds in fact for all $\phi \in H^2$, that is

$$\int \partial_t u^\tau(t) \psi = -\int \left(\partial_{xx} \overline{u}^\tau(t) + (\partial_x \overline{u}^\tau(t))^2\right) \psi_{xx}, \qquad \psi \in H^2, t \in [0,T). \tag{4.12}$$

Taking $\psi := \phi(t)$ for some $\phi \in C_0^\infty(\mathbb{T} \times [0,T))$ and integrating in time gives

$$\int_0^T \int \partial_t u^\tau \phi = -\int_0^T \int \left(\partial_{xx} \overline{u}^\tau + (\partial_x \overline{u}^\tau(t))^2 \right) \phi_{xx}, \qquad \phi \in C_0^\infty(\mathbb{T} \times [0,T)). \tag{4.13}$$

From here one can apply Hölder's inequality, the Sobolev embedding $H^{1/5} \subset L^{10/3}$, the Sobolev interpolation (4.3), (4.10) and a standard density argument to obtain a uniform (in τ) estimate on $\partial_t u^\tau$ in $L^{5/3}((0,T); (W^{2,5/2})^*)$. This, the energy inequalities (4.10), (4.11), and the Aubin–Lions lemma (see Theorem 2.1 in Section 3.2 in Temam, 1977, for example) give the existence of a sequence $\tau_n \to 0^+$ and a $u \in L^2((0,T); W^{1,\infty})$ such that

$$\begin{aligned} u^{\tau_n} &\to u && \text{in } L^2((0,T); W^{1,\infty}), \\ \overline{u}^{\tau_n}, u^{\tau_n} &\rightharpoonup u && \text{in } L^2((0,T); H^2), \\ \overline{u}^{\tau_n}, u^{\tau_n} &\overset{*}{\rightharpoonup} u && \text{in } L^\infty((0,T); L^2) \end{aligned} \tag{4.14}$$

as $\tau_n \to 0$. Here "$\rightharpoonup$" and "$\overset{*}{\rightharpoonup}$" denote the weak and weak-$*$ convergence, respectively. The fact that both $\overline{u}^{\tau_n}$ and u^{τ_n} converge to the same limit function follows from the convergence

$$\| u^{\tau_n} - \overline{u}^{\tau_n} \|_{L^2((0,T);W^{1,\infty})} \to 0 \quad \text{ as } \tau_n \to 0,$$

which can be shown using the first convergence from (4.14); see Lemma 2.3 in King et al. (2003) for details.

The limit function u is a weak solution to the surface growth initial value problem since the regularity requirement (4.5) follows from the convergence above and (4.6) follows by taking the limit $\tau_n \to 0^+$ in (4.13) after integration by parts in time of the left-hand side. □

As with the partial regularity theory for the Navier–Stokes equations, we make key use of a local energy inequality. This gives rise to the notion of "suitable weak solutions", which we now define.

Definition 4.3 (Suitable weak solution)**.** *We say that a weak solution is* suitable *if the local energy inequality*

$$\begin{aligned} \frac{1}{2} \int u(t)^2 \phi(t) + \int_0^t \int u_{xx}^2 \phi \le \int_0^t \int \bigg(& \frac{1}{2} (\phi_t - \phi_{xxxx}) u^2 \\ & + 2 u_x^2 \phi_{xx} - \frac{5}{3} u_x^3 \phi_x - u_x^2 u \phi_{xx} \bigg) \end{aligned} \tag{4.15}$$

holds for all $\phi \in C_0^\infty(\mathbb{T} \times (0,\infty); [0,\infty))$ and almost all $t \in (0,T)$.

Note that the local energy inequality is a weak form of the inequality

$$u\left(u_t + u_{xxxx} + \partial_{xx} u_x^2\right) \le 0;$$

that is (4.15) can be obtained (formally) by multiplying the above inequality by ϕ and integrating by parts. We note that (4.15) remains true if u is replaced by $u - K$ for any $K \in \mathbb{R}$. Indeed, multiplying (4.7) with $s := 0$ by K (and integrating by parts the term with four x derivatives) we obtain

$$-K \int u(t)\phi(t) = -K \int_0^t \int \left(u\phi_t - u\phi_{xxxx} - u_x^2 \phi_{xx}\right).$$

Thus noting that

$$\frac{K^2}{2} \int \phi(t) = \int_0^t \int \frac{1}{2}\left(\phi_t - \phi_{xxxx}\right) K^2$$

we obtain the claim by adding the above two equalities from (4.15).

By adapting the method outlined above in the proof of the existence of a weak solution, we now show that this solution also satisfies the local energy inequality and is therefore ‘suitable’.

Theorem 4.4. *The weak solution given by Theorem 4.2 is suitable.*

Proof. Fix $\phi \in C_0^\infty(\mathbb{T} \times (0,T))$ with $\phi \ge 0$. We will show that

$$\int_0^T \int u_{xx}^2 \phi \le \int_0^T \int \left(\frac{1}{2}(\phi_t - \phi_{xxxx})u^2 + 2u_x^2\phi_{xx} - \frac{5}{3}u_x^3\phi_x - u_x^2 u\phi_{xx}\right). \tag{4.16}$$

This is equivalent to (4.15), which can be shown using a cut-off procedure (in time), similarly to the equivalence between (4.6) and (4.7).

Let n be large enough so that $\phi(t) \equiv 0$ for $t \in (0, 2\tau_n) \cup (T - 2\tau_n, T)$. For brevity we will write τ in place of τ_n. Given $t \in [0,T)$ set $\varphi := \phi(t)$ and let k be such that $t \in [(k-1)\tau, k\tau)$. Let $\psi := u_k^\tau \varphi$ in (4.12) to obtain

$$\int \frac{u_k^\tau - u_{k-1}^\tau}{\tau} u_k^\tau \varphi = -\int \partial_{xx} u_k^\tau \left(u_k^\tau \varphi\right)_{xx} - \int \left(\partial_x u_k^\tau\right)^2 \left(u_k^\tau \varphi\right)_{xx}. \tag{4.17}$$

Since integration by parts gives for any $v \in H^2$

$$\int v_{xx} v_x \varphi_x = -\frac{1}{2} \int v_x^2 \varphi_{xx},$$

$$\int v_{xx} v \varphi_{xx} = -\int v_x^2 \varphi_{xx} - \int v_x v \varphi_{xxx} = -\int v_x^2 \varphi_{xx} + \frac{1}{2} \int v^2 \varphi_{xxxx},$$

$$\int v_x^2 v_{xx} \varphi = -\frac{1}{3} \int v_x^3 \varphi_x,$$

the first term on the right-hand side of (4.17) can be written in the form

$$\begin{aligned}
-\int \partial_{xx}u_k^\tau \,(u_k^\tau\varphi)_{xx} &= -\int (\partial_{xx}u_k^\tau)^2\phi - 2\int \partial_{xx}u_k^\tau \partial_x u_k^\tau \varphi_x - \int \partial_{xx}u_k^\tau \, u_k^\tau \varphi_{xx} \\
&= -\int (\partial_{xx}u_k^\tau)^2\phi + 2\int (\partial_x u_k^\tau)^2 \varphi_{xx} - \frac{1}{2}\int (u_k^\tau)^2 \varphi_{xxxx}.
\end{aligned}$$

Similarly, the second term in (4.17) can be expanded into

$$\begin{aligned}
-\int (\partial_x u_k^\tau)^2 \,(u_k^\tau\varphi)_{xx} &= -\int (\partial_x u_k^\tau)^2 \,\partial_{xx}u_k^\tau\varphi - 2\int (\partial_x u_k^\tau)^2\,\partial_x u_k^\tau \varphi_x \\
&\quad - \int (\partial_x u_k^\tau)^2\, u_k^\tau \varphi_{xx} \\
&= -\frac{5}{3}\int (\partial_x u_k^\tau)^3\,\varphi_x - \int (\partial_x u_k^\tau)^2\, u_k^\tau\varphi_{xx}.
\end{aligned}$$

On the other hand, using the inequality $ab \le a^2/2 + b^2/2$ we can bound the left-hand side of (4.17) from below by writing

$$\begin{aligned}
\int \frac{u_k^\tau - u_{k-1}^\tau}{\tau} u_k^\tau\varphi &= \frac{1}{\tau}\|u_k^\tau\sqrt{\varphi}\|^2 - \frac{1}{\tau}\int u_k^\tau\sqrt{\varphi}\, u_{k-1}^\tau\sqrt{\varphi} \\
&\ge \frac{1}{2\tau}\|u_k^\tau\sqrt{\varphi}\|^2 - \frac{1}{2\tau}\|u_{k-1}^\tau\sqrt{\varphi}\|^2.
\end{aligned}$$

Substituting these calculations into (4.17) gives

$$\begin{aligned}
&\frac{1}{2\tau}\|u_k^\tau\sqrt{\varphi}\|^2 - \frac{1}{2\tau}\|u_{k-1}^\tau\sqrt{\varphi}\|^2 + \int (\partial_{xx}u_k^\tau)^2\varphi \\
&\quad\le \int \left(2(\partial_x u_k^\tau)^2\varphi_{xx} - \frac{1}{2}(u_k^\tau)^2\varphi_{xxxx} - \frac{5}{3}(\partial_x u_k^\tau)^3\varphi_x - (\partial_x u_k^\tau)^2 u_k^\tau\varphi_{xx} \right).
\end{aligned}$$

Integration in time gives

$$\begin{aligned}
&\frac{1}{2\tau}\int_0^T \|\overline{u}^\tau(t)\sqrt{\phi(t)}\|^2 \mathrm{d}t - \frac{1}{2\tau}\int_0^T \|\overline{u}^\tau(t-\tau)\sqrt{\phi(t)}\|^2\mathrm{d}t + \int_0^T\int (\overline{u}^\tau_{xx})^2\phi \\
&\quad\le \int_0^T\int \left(2(\overline{u}^\tau_x)^2\phi_{xx} - \frac{1}{2}(\overline{u}^\tau)^2\phi_{xxxx} - \frac{5}{3}(\overline{u}^\tau_x)^3\phi_x - (\overline{u}^\tau_x)^2\,\overline{u}^\tau\phi_{xx} \right).
\end{aligned} \tag{4.18}$$

Observe that the convergence $\overline{u}^\tau \to u$ in $L^2((0,T);W^{1,\infty})$ (see (4.14)) gives the convergence of the right-hand side above to the respective expression with u,

$$\int_0^T\int \left(2u_x^2\phi_{xx} - \frac{1}{2}u^2\phi_{xxxx} - \frac{5}{3}u_x^3\phi_x - u_x^2 u\phi_{xx} \right).$$

Moreover, the weak convergence $\overline{u}^\tau \rightharpoonup u$ in $L^2((0,T);H^2)$ (see (4.14)) gives in particular the weak convergence

$$\overline{u}^\tau_{xx}\sqrt{\phi} \rightharpoonup u_{xx}\sqrt{\phi} \quad \text{in } L^2((0,T);L^2) \text{ as } \tau\to 0,$$

and thus, from properties of weak limits,

$$\int_0^T \int u_{xx}^2 \phi \leq \liminf_{\tau\to 0} \int_0^T \int \left(\overline{u}_{xx}^\tau\right)^2 \phi.$$

As for the first two terms in (4.18), they can be written in the form

$$\begin{aligned}\int_0^T \frac{\|\overline{u}^\tau(t)\sqrt{\phi(t)}\|^2 - \|\overline{u}^\tau(t-\tau)\sqrt{\phi(t-\tau)}\|^2}{2\tau} \mathrm{d}t \\ - \frac{1}{2}\int_0^T \left((\overline{u}^\tau(t-\tau))^2, \frac{\phi(t)-\phi(t-\tau)}{\tau} \right) \mathrm{d}t,\end{aligned} \tag{4.19}$$

where $(\cdot,\cdot)$ denotes the L^2 product. Observe that the first term vanishes due to the change of variable $t' := t-\tau$ and the fact that ϕ vanishes on time intervals $(0,2\tau)$ and $(T-2\tau,T)$. A similar change of variables in the second term gives that (4.19) equals

$$-\frac{1}{2}\int_0^T \left((\overline{u}^\tau(t))^2, \frac{\phi(t+\tau)-\phi(t)}{\tau} \right) \mathrm{d}t.$$

Thus the convergence $\overline{u}^\tau \to u$ in $L^2((0,T);W^{1,\infty})$ and the fact that

$$\frac{\phi(x,t+\tau)-\phi(x,t)}{\tau} \to \phi_t(x,t) \qquad \text{uniformly in } (x,t)\in\mathbb{T}\times(0,T)$$

give that (4.19) converges to

$$-\frac{1}{2}\int_0^T \int u^2 \phi_t$$

as $\tau \to 0^+$. Hence, altogether, taking $\liminf_{\tau\to 0^+}$ (recall we write τ in place of τ_n) in (4.18) gives the local energy inequality (4.16), as required. □

4.3 A 'nonlinear' parabolic Poincaré inequality

Here we prove a parabolic version of the Poincaré inequality, which is a key ingredient in the proof of the partial regularity results that follow.

Theorem 4.5 (Parabolic Poincaré inequality)**.** *Let $\eta\in[0,1]$, $r\in(0,1)$ and let $Q(z_0,r)$ be a cylinder, where $z_0=(x_0,t_0)$. If a function u satisfies*

$$\int_{B_r(x_0)} (u(t)-u(s))\phi = \int_s^t \int_{B_r(x_0)} u_x\phi_{xxx} - \eta\int_s^t\int_{B_r(x_0)} u_x^2\phi_{xx} \tag{4.20}$$

for all $\phi\in C_0^\infty(B_r(x_0))$ and almost every $s,t\in(-r^4,r^4)$ with $s<t$, then

$$\frac{1}{r^5}\int_{Q(z_0,r/2)} |u-u_{z_0,r/2}|^3 \leq c_{pp}\left(\mathrm{Y}(z_0,r)+\eta\mathrm{Y}(z_0,r)^2\right), \tag{4.21}$$

where

$$\Upsilon(z_0, r) := \frac{1}{r^2} \int_{Q(z_0,r)} |u_x|^3 \tag{4.22}$$

and $c_{pp} > 0$ *is an absolute constant.*

Recall $u_{z_0,r/2}$ denotes the mean of u over $Q(z_0, r/2)$ (see (4.2)). Note that no t derivative appears on the right-hand side of (4.21). Observe that (4.21) is the classical Poincaré inequality if $\eta = 0$ and the left-hand side is replaced by

$$\frac{1}{r^5} \int_{Q(z_0,r/2)} \left| u - \fint_{B(x_0,r/2)} u(t) \right| \mathrm{d}x \, \mathrm{d}t$$

(i.e. the mean over the cylinder is replaced by the mean over the ball at each time). Moreover note that (4.21) does not hold for arbitrary functions since adding a function of time to u allows one to increase the left-hand side while keeping the right-hand side bounded. This also verifies the relevance of the assumption (4.20) since it shows that the only function of time which can be added to u is a constant function. On the other hand, adding constants to u makes no change to (4.21).

Furthermore, the case $\eta = 0$ gives the parabolic Poincaré inequality for weak solutions to the biharmonic heat equation:

$$\frac{1}{r^3} \int_{Q(z_0,r/2)} |u - u_{z_0,r/2}|^3 \le c_{pp} \int_{Q(z_0,r)} |u_x|^3,$$

whenever $u_t = \partial_x^4 u$ (weakly). In this case it can be shown that the inequality holds in any dimension (with ∂_x^4 replaced by the bilaplacian Δ^2) and for any $p \ge 1$.

Due to (4.7) any weak solution of the surface growth equation satisfies (4.20) for all z_0, r as long as $Q(z_0, r) \subset \mathbb{T} \times (0, T)$, and hence we can use inequality (4.21) for the suitable weak solutions that form our main subject in what follows.

We prove this nonlinear parabolic Poincaré inequality adapting the approach used by Aramaki (2016) in the context of the heat equation, itself based on previous work by Struwe (1981).

Proof. Fix r and z_0 and set, for brevity

$$Q_\rho := Q(z_0, \rho), \quad B_\rho := B(x_0, \rho) \qquad \text{for } \rho > 0, \text{ where } z_0 = (x_0, t_0),$$

and set

$$M := \Upsilon(z_0, r).$$

Step 1. We introduce the notion of σ-means.

Let $\sigma\colon \mathbb{R} \to [0,1]$ be the cut-off function in space around x_0 such that

$$\sigma(x) = \begin{cases} 1 & |x-x_0| \le r/2, \\ 0 & |x-x_0| \ge r, \end{cases} \qquad |\partial_x^k \sigma| \le C r^{-k},\ k \ge 0.$$

Let

$$u_r^\sigma(t) := \frac{\int_{B_r} u(t)\sigma \,\mathrm{d}x}{\int_{B_r} \sigma \,\mathrm{d}x}, \qquad [u]_r^\sigma := \frac{\int_{Q_r} u\sigma \,\mathrm{d}z}{\int_{Q_r} \sigma \,\mathrm{d}z} \tag{4.23}$$

denote the σ*-mean* of u over a ball (at a given time t) and over a cylinder, respectively. Note that, since σ is a function of x only,

$$u_r^\sigma(t) - [u]_r^\sigma = \frac{1}{2r^4} \int_{-r^4}^{r^4} \left(u_r^\sigma(t) - u_r^\sigma(s) \right) \mathrm{d}s. \tag{4.24}$$

Furthermore, let us write for brevity

$$u_r := u_{z_0,r};$$

then

$$\int_{Q_{r/2}} |u - u_{r/2}|^3 \le 8 \int_{Q_{r/2}} |u - [u]_r^\sigma|^3. \tag{4.25}$$

Indeed, by writing

$$|u_{r/2} - L|^3 = \left| \frac{1}{|Q_{r/2}|} \int_{Q_{r/2}} u - L \right|^3 \le \frac{1}{|Q_{r/2}|} \int_{Q_{r/2}} |u - L|^3 ,$$

where $L := [u]_r^\sigma$, we see that the triangle inequality gives

$$\begin{aligned} \left(\int_{Q_{r/2}} |u - u_{r/2}|^3 \right)^{1/3} &\le \left(\int_{Q_{r/2}} |u - L|^3 \right)^{1/3} + \left(\int_{Q_{r/2}} |u_{r/2} - L|^3 \right)^{1/3} \\ &\le 2 \left(\int_{Q_{r/2}} |u - L|^3 \right)^{1/3}, \end{aligned}$$

as required. In what follows we will also use the following classical Poincaré inequality: for $t \in (0,T)$, $q \ge 1$, $r \in (0,1)$,

$$\int_{B_r} |u(t) - u_r^\sigma(t)|^q \sigma \le C(n,q) r^q \int_{B_r} |u_x(t)|^q \sigma, \tag{4.26}$$

see Lemma 6.12 in Lieberman Lieberman (2005) for a proof.

Step 2. We show that for almost every $s, t \in (-r^4, r^4)$

$$|u_r^\sigma(t) - u_r^\sigma(s)|^3 \leq C(M + \eta M^2). \tag{4.27}$$

To this end suppose (without loss of generality) that $s < t$ and let

$$\phi(x) := \sigma(x)(u_r^\sigma(t) - u_r^\sigma(s))|u_r^\sigma(t) - u_r^\sigma(s)|,$$

be the test function in (4.20). Then the term on the left-hand side can be bounded from below,

$$\begin{aligned}\int_{B_r}(u(t)-u(s))\phi &= (u_r^\sigma(t) - u_r^\sigma(s))|u_r^\sigma(t) - u_r^\sigma(s)| \int_{B_r}(u(t)-u(s))\sigma \\ &= |u_r^\sigma(t) - u_r^\sigma(s)|^3 \int_{B_r}\sigma \geq Cr|u_r^\sigma(t) - u_r^\sigma(s)|^3.\end{aligned}$$

The first term on the right-hand side can be estimated by writing

$$\begin{aligned}\left|\int_s^t \int_{B_r} u_x \phi_{xxx}\right| &\leq |u_r^\sigma(t) - u_r^\sigma(s)|^2 \int_s^t \int_{B_r} |u_x|\,|\sigma_{xxx}| \\ &\leq C|u_r^\sigma(t) - u_r^\sigma(s)|^2 r^{-3}\int_{Q_r}|u_x| \\ &\leq C|u_r^\sigma(t) - u_r^\sigma(s)|^2 r^{-3}\left(\int_{Q_r}|u_x|^3\right)^{1/3} r^{10/3} \\ &\leq \delta r|u_r^\sigma(t) - u_r^\sigma(s)|^3 + C_\delta r^{-1}\int_{Q_r}|u_x|^3 \\ &= \delta r|u_r^\sigma(t) - u_r^\sigma(s)|^3 + rMC_\delta\end{aligned}$$

for any $\delta > 0$, where we used Hölder's inequality and Young's inequality in the form

$$a^2 r^{1/3} b^{1/3} \leq \delta a^3 r + C_\delta b r^{-1}.$$

The second term on the right-hand side can be estimated by writing

$$\begin{aligned}\left|\int_s^t \int_{B_r} u_x^2 \phi_{xx}\right| &\leq |u_r^\sigma(t) - u_r^\sigma(s)|^2 \int_s^t \int_{B_r} |u_x|^2\,|\sigma_{xx}| \\ &\leq Cr^{-2}|u_r^\sigma(t) - u_r^\sigma(s)|^2 \int_{Q_r}|u_x|^2 \\ &\leq Cr^{-2}|u_r^\sigma(t) - u_r^\sigma(s)|^2\left(\int_{Q_r}|u_x|^3\right)^{2/3} r^{5/3} \\ &\leq \delta r|u_r^\sigma(t) - u_r^\sigma(s)|^3 + C_\delta r^{-3}\left(\int_{Q_r}|u_x|^3\right)^2 \\ &= \delta r|u_r^\sigma(t) - u_r^\sigma(s)|^3 + C_\delta r M^2, \qquad \delta > 0,\end{aligned}$$

where we used Hölder's inequality and Young's inequality in the form

$$a^2 r^{-1/3} b^{2/3} \le \delta r a^3 + C_\delta r^{-3} b^2.$$

Since $\eta \le 1$ (see (4.20)) we therefore obtain

$$Cr|u_r^\sigma(t) - u_r^\sigma(s)|^3 \le 2\delta r|u_r^\sigma(t) - u_r^\sigma(s)|^3 + rC_\delta(M + \eta M^2),$$

and fixing $\delta > 0$ sufficiently small gives (4.27).

Step 3. We show (4.21).

From (4.25), the fact that $\sigma \in [0,1]$ with $\sigma = 1$ on $Q_{r/2}$ and the inequality $\int |f + g|^q \le 2^q \int |f|^q + 2^q \int |g|^q$ we obtain

$$\begin{aligned}\int_{Q_{r/2}} |u - u_{r/2}|^3 &\le 8 \int_{Q_r} |u - [u]_r^\sigma|^3 \sigma \,\mathrm{d}x\,\mathrm{d}t \\ &\le 64 \int_{Q_r} |u - u_r^\sigma(t)|^3 \sigma \,\mathrm{d}x\,\mathrm{d}t + 64 \int_{Q_r} |u_r^\sigma(t) - [u]_r^\sigma|^3 \mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{4.28}$$

The first of the resulting integrals can be bounded using (4.26),

$$\int_{Q_r} |u - u_r^\sigma(t)|^3 \sigma \,\mathrm{d}x\,\mathrm{d}t \le Cr^3 \int_{Q_r} |u_x|^3 \sigma \le Cr^5 M.$$

The second one can be bounded using (4.24) and Step 2,

$$|u_r^\sigma(t) - [u]_r^\sigma|^3 \le \frac{1}{2r^4} \int_{-r^4}^{r^4} |u_r^\sigma(t) - u_r^\sigma(s)|^3 \,\mathrm{d}s \le C(M + \eta M^2), \tag{4.29}$$

which gives

$$\int_{Q_r} |u_r^\sigma(t) - [u]_r^\sigma|^3 \mathrm{d}x\,\mathrm{d}t \le Cr^5(M + \eta M^2)$$

Applying these bounds in (4.28) gives

$$\int_{Q_{r/2}} |u - u_{r/2}|^3 \le Cr^5(M + \eta M^2),$$

that is (4.21). □

Corollary 4.6. *The claim of Theorem 4.5 remains valid if* (4.21) *is replaced by*

$$\frac{1}{r^5} \int_{Q(z_0,r)} |u - [u]_r^\sigma|^3 \,\sigma \;\le c\left(\Upsilon(z_0,r) + \eta \Upsilon(z_0,r)^2\right),$$

Proof. This follows by ignoring the first inequality in (4.28). □

4.4 The first conditional and partial regularity results

Here we show local regularity of suitable weak solutions to the surface growth equation based on a condition on u_x. Namely, we will show in Theorem 4.12 that there exists $\varepsilon_0 > 0$ and $R_0 > 0$ such that if

$$\frac{1}{r^2}\int_{Q(z,r)} |u_x|^3 < \varepsilon_0$$

for some $r < R_0$ and z then u is Hölder continuous in $Q(z, r/2)$.

The proof we give of this result is based on that presented for the Navier–Stokes equations by Ladyzhenskaya & Seregin (1999); we begin with a certain 'one-step' decay estimate, which we then iterate.

4.4.1 Interior regularity for the biharmonic heat flow

The proof of the decay estimate relies on the following regularity result for the biharmonic heat equation; while the result is perhaps 'standard', I could not find an obvious canonical reference, and so for the sake of completeness I provide a short proof.

Proposition 4.7 (Interior regularity of the biharmonic heat flow)**.** *Suppose that* $0 < b < a$*,* $v, v_x \in L^2(Q_a)$ *and that* v *is a distributional solution to the biharmonic heat equation* $v_t = -v_{xxxx}$ *in* Q_a*, that is*

$$\iint_{Q_a} v\,\phi_t = \iint_{Q_a} v\,\phi_{xxxx} \tag{4.30}$$

for every $\phi \in C_0^\infty(Q_a)$*. Then*

$$\|v_x\|_{L^\infty(Q_b)} \le C_{a,b}\left(\|v\|_{L^2(Q_a)} + \|v_x\|_{L^2(Q_a)}\right)$$

for some $C_{a,b} > 0$*.*

Proof. We assume that $a = 1$, $b = 1/2$; the claim for arbitrary a, b follows similarly. First we show that $v_{xx} \in L^2(Q_{7/8})$ with

$$\|v_{xx}\|_{L^2(Q_{7/8})} \le C\left(\|v\|_{L^2(Q_1)} + \|v_x\|_{L^2(Q_1)}\right). \tag{4.31}$$

For this let $\varepsilon \in (0, 1/16)$. Then $\phi^{(\varepsilon)} \in C_0^\infty(Q_1)$ for every $\phi \in C_0^\infty(Q_{15/16})$, where $\phi^{(\varepsilon)}$ denotes the standard mollification (in both space an time) of ϕ. Using $\phi^{(\varepsilon)}$ as a test function in (4.30) and applying the Fubini Theorem we obtain

$$\iint_{Q_{15/16}} v^{(\varepsilon)}\,\phi_t = \iint_{Q_{15/16}} v^{(\varepsilon)}\,\phi_{xxxx}, \qquad \phi \in C_0^\infty(Q_{15/16}),$$

that is $v^{(\varepsilon)}$ is a distributional solution of the biharmonic heat equation in $Q_{15/16}$. Moreover, from properties of mollification,

$$\|v^{(\varepsilon)}\|_{L^2(Q_{15/16})} \leq \|v\|_{L^2(Q_1)} \qquad \text{and} \qquad \|v_x^{(\varepsilon)}\|_{L^2(Q_{15/16})} \leq \|v_x\|_{L^2(Q_1)} \tag{4.32}$$

for all ε. Since $v^{(\varepsilon)}$ is smooth it satisfies the equation

$$v_t^{(\varepsilon)} = -v_{xxxx}^{(\varepsilon)}$$

in the classical sense. Multiplying this equation by $v^{(\varepsilon)}\phi$ (where $\phi \in C_0^\infty(Q_{15/16})$) and integrating by parts on $Q_{15/16}$ gives

$$\iint_{Q_{15/16}} (v_{xx}^{(\varepsilon)})^2 \phi = \iint_{Q_{15/16}} \left(\frac{1}{2} (v^{(\varepsilon)})^2 (\phi_t - \phi_{xxxx}) + 2 (v_x^{(\varepsilon)})^2 \phi_{xx} \right)$$

for every $\phi \in C_0^\infty(Q_{15/16})$. Taking $\phi \geq 0$ such that $\phi = 1$ on $Q_{7/8}$ we obtain

$$\|v_{xx}^{(\varepsilon)}\|_{L^2(Q_{7/8})} \leq C_\rho \left(\|v\|_{L^2(Q_1)} + \|v_x\|_{L^2(Q_1)} \right),$$

where we used (4.32). Thus $v_{xx}^{(\varepsilon)}$ is a bounded in $L^2(Q_{7/8})$ and hence there exists a sequence $\varepsilon_k \to 0^+$ such that $v_{xx}^{(\varepsilon)} \rightharpoonup v_{xx}$ weakly in $L^2(Q_{7/8})$. Note that the limit function is v_{xx} by definition of weak derivatives since $v^{(\varepsilon)} \to v$ strongly in $L^2(Q_{15/16})$. Thus in particular $v_{xx} \in L^2(Q_{7/8})$ and, using a property of weak limits and the last inequality, we obtain

$$\|v_{xx}\|_{L^2(Q_{7/8})} \leq \liminf_{\varepsilon_k \to 0^+} \|v_{xx}^{(\varepsilon_k)}\|_{L^2(Q_{7/8})} \leq C_\rho \left(\|v\|_{L^2(Q_1)} + \|v_x\|_{L^2(Q_1)} \right),$$

that is (4.31), as required.

Now letting $\phi \coloneqq \psi_x$ for some $\psi \in C_0^\infty(Q_{7/8})$ we see from (4.30) that v_x is a distributional solution of the biharmonic heat equation in $Q_{7/8}$. Moreover, using (4.31), we see that $v_x, v_{xx} \in L^2(Q_{7/8})$. Thus applying a similar argument as in the case of (4.31) we obtain that $v_{xxx} \in L^2(Q_{3/4})$ with

$$\|v_{xxx}\|_{L^2(Q_{3/4})} \leq C \left(\|v\|_{L^2(Q_1)} + \|v_x\|_{L^2(Q_1)} \right).$$

In the same way we observe that any spatial derivative of v is a distributional solution of the biharmonic heat equation, and $\partial_x^k v \in L^2(Q_{1/2})$ for all $k \leq 9$ with

$$\|\partial_x^k v\|_{L^2(Q_{1/2})} \leq C \left(\|v\|_{L^2(Q_1)} + \|v_x\|_{L^2(Q_1)} \right), \qquad k \leq 9.$$

Now since (4.30) gives in particular that $v_t = -v_{xxxx}$ in the sense of weak derivatives, we obtain from the above that each of $v_x, v_{xx}, v_{xt}, v_{xxx}, v_{xxt}, v_{xtt}$ is bounded in $L^2(Q_{1/2})$ by $C \left(\|v\|_{L^2(Q_1)} + \|v_x\|_{L^2(Q_1)} \right)$. Therefore the claim of the lemma follows from the two-dimensional embedding $H^2 \subset L^\infty$. $\square$

4.4.2 The 'one-step' estimate

Let u be a suitable weak solution of the surface growth model. In what follows we assume that a cylinder $Q(z,r)$ is contained in $\mathbb{T}\times(0,\infty)$, the domain of definition of u. We now state and prove the 'one-step' estimate.

Lemma 4.8. *Given $\theta\in(0,1/4)$ there exist $\varepsilon_*=\varepsilon_*(\theta)$ and $R=R(\theta)$ such that if $r<R$ and*

$$\mathrm{Y}(z,r):=\frac{1}{r^2}\int_{Q(z,r)}|u_x|^3<\varepsilon_*$$

then

$$\mathrm{Y}(z,\theta r)\le c_*\theta^3\mathrm{Y}(z,r), \tag{4.33}$$

where c_* is a universal constant.

Proof. We will show the claim for

$$c_*:=8C_{1/2,1/4}^3\left(1+c_{pp}^{1/3}\right)^3,$$

where $C_{1/2,1/4}$ is the constant from Proposition 4.7 and c_{pp} is from the parabolic Poincaré inequality (Theorem 4.5). Suppose that the claim is not true. Then there exist $r_k\to 0$, $\varepsilon_k\to 0$, and $z_k=(x_k,t_k)$ such that

$$\mathrm{Y}(z_k,r_k)=\frac{1}{r_k^2}\int_{Q(z_k,r_k)}|u_x|^3=\varepsilon_k,$$

but

$$\frac{1}{(\theta r_k)^2}\int_{Q(z_k,\theta r_k)}|u_x|^3\ge c_*\theta^3\varepsilon_k.$$

Step 1. We take a limit of rescaled solutions.

Let

$$u_k(x,t):=\frac{u\left(x_k+x\,r_k,t_k+t\,r_k^4\right)-u_{z_k,r_k/2}}{\varepsilon_k^{1/3}}$$

be a family of rescalings of u. Then $\{u_k\}$ is a family of functions such that $\int_{Q_{1/2}}u_k=0$ (which will be used shortly when we apply the parabolic Poincaré inequality),

$$\int_{Q_1}|\partial_x u_k|^3=1, \tag{4.34}$$

$$\int_{Q_\theta}|\partial_x u_k|^3\ge c_*\theta^5, \tag{4.35}$$

and u_k satisfies the local energy inequality

$$\begin{aligned}\int_{B_1} |u_k(t)|^2\phi(t) + \int_{-1}^t \int_{B_1} (\partial_{xx}u_k)^2\phi \le \int_{-1}^t \int_{B_1} \Big(\frac{1}{2}(\phi_t - \phi_{xxxx})(u_k)^2 \\ +2(\partial_x u_k)^2\phi_{xx} - \frac{5}{3}\varepsilon_k^{1/3}(\partial_x u_k)^3\phi_x - \varepsilon_k^{1/3}(\partial_x u_k)^2 u_k \phi_{xx} \Big)\end{aligned} \tag{4.36}$$

for all nonnegative $\phi \in C_0^\infty(Q_1)$ and almost all $t \in (-1,1)$ (recall (4.15)). Moreover u_k satisfies the equation $\partial_t u_k = -\partial_x^4 u_k - \varepsilon_k^{1/3}\partial_{xx}(\partial_x u_k)^2$ in Q_1 in the sense of distributions, that is

$$\iint_{Q_1} u_k\, \phi_t = \iint_{Q_1} u_k\, \phi_{xxxx} + \varepsilon_k^{1/3} \iint_{Q_1} (\partial_x u_k)^2\, \phi_{xx}, \quad \phi \in C_0^\infty(Q_1). \tag{4.37}$$

It follows from the parabolic Poincaré inequality (Theorem 4.5) and (4.34) that

$$\int_{Q_{1/2}} |u_k|^3 \le c_{pp}(1+\varepsilon_k^{1/3}). \tag{4.38}$$

Thus both u_k and $\partial_x u_k$ are bounded in $L^3(Q_{1/2})$ and hence there exists $v \in L^3(Q_{1/2})$ such that $\|v\|_{L^3(Q_{1/2})} \le c_{pp}^{1/3}$, $\|v_x\|_{L^3(Q_{1/2})} \le 1$ and

$$u_{k_n} \rightharpoonup v, \quad \partial_x u_{k_n} \rightharpoonup v_x \qquad \text{in } L^3(Q_{1/2}) \text{ as } n \to \infty$$

for some sequence $k_n \to \infty$. Taking the limit in (4.37) we obtain

$$\iint_{Q_{1/2}} v\, \phi_t = \iint_{Q_{1/2}} v\, \phi_{xxxx}, \quad \phi \in C_0^\infty(Q_{1/2}),$$

that is the limit function v is a distributional solution of the biharmonic heat equation $v_t = -v_{xxxx}$ on $Q_{1/2}$. In particular, using Proposition 4.7, we obtain

$$\begin{aligned}\|v_x\|_{L^\infty(Q_{1/4})} &\le C_{1/2,1/4}\left(\|v\|_{L^2(Q_{1/2})} + \|v_x\|_{L^2(Q_{1/2})} \right) \\ &\le C_{1/2,1/4}(1+c_{pp}^{1/3}) = (c_*/8)^{1/3}.\end{aligned} \tag{4.39}$$

Step 2. We show strong convergence $\partial_x u_{k_n} \to v_x$ in $L^3(Q_{1/4})$ on a subsequence k_n (relabelled).

We will write $k \coloneqq k_n$ for brevity. Letting $\phi \in C_0^\infty(Q_{1/2})$ be nonnegative and such that $\phi = 1$ on $Q_{1/4}$ the local energy inequality (4.36) gives

$$\|u_k(t)\|^2_{L^2(B_{1/4})} + \int_{-4^{-4}}^t \|\partial_{xx}u_k(s)\|^2_{L^2(B_{1/4})}\mathrm{d}s \le C$$

for almost every $t \in (-4^{-4}, 4^{-4}) =: I_{1/4}$, where we also used (4.38), (4.34) and the fact that $\varepsilon_k < 1$, and thus

$$\|u_k\|_{L^\infty(I_{1/4};L^2(B_{1/4}))} + \|\partial_{xx} u_k\|_{L^2(Q_{1/4})} \leq C. \tag{4.40}$$

Using 1D Sobolev interpolation $\|v\|_{H^{4/3}} \leq \|v\|_{L^2}^{1/3} \|v\|_{H^2}^{2/3}$ (recall (4.3)) this in particular gives

$$\|u_k\|_{L^3(I_{1/4};H^{4/3}(B_{1/4}))} \leq C. \tag{4.41}$$

Moreover, from (4.37) we obtain

$$\begin{aligned}\left| \iint_{Q_{1/4}} \partial_t u_k \, \phi \right| &= \left| - \iint_{Q_{1/4}} \partial_{xx} u_k \, \phi_{xx} - \varepsilon_k^{1/3} \iint_{Q_{1/4}} (\partial_x u_k)^2 \, \phi_{xx} \right| \\ &\leq \|\phi\|_{L^3(I_{1/4};W^{2,3}(B_{1/4}))} \left(\|\partial_{xx} u_k\|_{L^{3/2}(Q_{1/4})} + \|\partial_x u_k\|^2_{L^3(Q_{1/4})} \right) \\ &\leq C \|\phi\|_{L^3(I_{1/4};W^{2,3}(B_{1/4}))}\end{aligned}$$

for all $\phi \in C_0^\infty(Q_{1/4})$, where the last inequality follows from Hölder's inequality, the bound (4.40) above and (4.34). By the density of $C_0^\infty(Q_{1/4})$ in $L^3(I_{1/4}; W^{2,3}(B_{1/4}))$ the above inequality gives boundedness of $\partial_t u_k$ in $L^{3/2}(I_{1/4}; (W^{2,3}(B_{1/4}))^*)$. This and (4.41) let us use the Aubin–Lions compactness lemma (see, for example, Section 3.2.2 in Temam, 2001) to extract a subsequence of (u_k) (which we relabel) that converges in $L^3(I_{1/4}; H^{7/6}(B_{1/4}))$. Using the 1D Sobolev embedding $H^{1/6} \subset L^3$ this in particular means that $\partial_x u_k$ converges in $L^3(Q_{1/4})$, as required.

Step 3. We use (4.35) to obtain a contradiction.

Since $\theta \in (0, 1/4)$ the last step gives in particular $\partial_x u_{k_n} \to v_x$ in $L^3(Q_\theta)$. Thus taking the limit $k_n \to \infty$ in (4.35) and using the L^∞ bound on v_x from (4.39) we obtain

$$1 \leq \frac{1}{c_* \theta^5} \int_{Q_\theta} |v_x|^3 \leq \frac{1}{8\theta^5} |Q_\theta| = \frac{1}{2},$$

a contradiction. □

4.4.3 Conditional regularity in terms of u_x

We now iterate this estimate.

Lemma 4.9. *Given $\alpha \in (0,3)$ there exist $\varepsilon_* > 0$ and $R \in (0,1)$ such that if $r < R$ and*

$$\frac{1}{r^2}\int_{Q(z,r)} |u_x|^3 < \varepsilon_* \tag{4.42}$$

then

$$\frac{1}{\varrho^2}\int_{Q(z,\varrho)} |u_x|^3 \le C\varepsilon_* \left(\frac{\varrho}{r}\right)^\alpha \qquad \textit{for all} \quad \varrho \le r. \tag{4.43}$$

Proof. Similarly as before we will use the notation $\Upsilon(z,r) = \frac{1}{r^2}\int_{Q(z,r)} |u_x|^3$. Fix $\theta \in (0,1/2)$ sufficiently small such that

$$c_*\theta^3 < \theta^\alpha.$$

Lemma 4.8 then guarantees that if $\Upsilon(z,r) < \varepsilon_*$ for some $r < R$ then

$$\Upsilon(z,\theta r) \le \theta^\alpha\, \Upsilon(z,r).$$

Iterating this result we obtain

$$\Upsilon(z,\theta^k r) \le \theta^{\alpha k}\Upsilon(z,r), \qquad k \ge 0.$$

Now for $\varrho \in (0,r)$ choose k such that

$$\theta^{k+1} r < \varrho \le \theta^k r;$$

then

$$\begin{aligned}
\Upsilon(z,\varrho) &= \frac{1}{\varrho^2}\int_{Q(z,\varrho)} |u_x|^3 \\
&\le \frac{1}{(\theta^{(k+1)}r)^2}\int_{Q(z,\theta^k r)} |u_x|^3 = \theta^{-2}\Upsilon(z,\theta^k r) \\
&\le \theta^{\alpha k - 2}\Upsilon(z,r) \\
&\le \theta^{-\alpha-2}\frac{\varrho}{r}\Upsilon(z,r),
\end{aligned}$$

which yields (4.43). □

Combining this decay estimate with the nonlinear parabolic Poincaré inequality (Theorem 4.5) yields the following.

Corollary 4.10. *Given $\alpha \in (0,3)$ there exist $\varepsilon_* > 0$ and $R \in (0,1)$ such that if $r < R$ and*

$$\frac{1}{r^2}\int_{Q(z,r)} |u_x|^3 < \varepsilon_*$$

then

$$\frac{1}{\varrho^5}\int_{Q(z,\varrho)} |u - u_{z,\varrho}|^3 \le C\varepsilon_* \left(\frac{\varrho}{r}\right)^\alpha \qquad \textit{for all} \quad \varrho \le r.$$

Our first conditional regularity result can be shown using the parabolic Campanato Lemma.

Lemma 4.11 (Parabolic Campanato Lemma). *Let $R \in (0,1)$, $f \in L^1(Q_R(0))$ and suppose that there exist positive constants $\beta \in (0,1]$, $M > 0$, such that*

$$\left(\fint_{Q_r(z)} |f(y) - f_{z,r}|^p \, \mathrm{d}y \right)^{1/p} \leq M r^\beta$$

for any $z \in Q_{R/2}(0)$ and any $r \in (0, R/2)$. Then f is Hölder continuous in $Q_{R/2}(0)$: for any $z, w \in Q_{R/2}(0)$, $z = (x,t)$, $w = (y,s)$,

$$|f(x,t) - f(y,s)| \leq cM(|x-y| + |t-s|^{1/4})^\beta.$$

The Campanato lemma is a standard tool, and we refer the reader to the Appendix in Ożański & Robinson (2017) for a proof.

The first conditional regularity result now follows.

Theorem 4.12 (Conditional regularity in terms of u_x). *Given $\beta \in (0,1)$ there exist $\varepsilon_0 > 0$ and $R_0 \in (0,1)$ such that if $r < R_0$ and*

$$\frac{1}{r^2} \int_{Q(z,r)} |u_x|^3 < \varepsilon_0 \tag{4.44}$$

then u is Hölder continuous in $Q(z, r/2)$, with

$$|u(x_1,t_1) - u(x_2,t_2)| \leq \frac{C}{r} \left(|x_1 - x_2| + |t_1 - t_2|^{1/4} \right)^\beta . \tag{4.45}$$

Proof. Let $\varepsilon_0 \coloneqq \varepsilon_*/4$, $R_0 \coloneqq \min\{1, R\}$ and $r < R_0$, where ε_*, R are from Corollary 4.10 applied with $\alpha = 3\beta$. Then $Q(y, r/2) \subset Q(z,r)$ for every $y \in Q(z, r/2)$ and

$$\frac{1}{(r/2)^2} \int_{Q(y,r/2)} |u_x|^3 \leq \frac{4}{r^2} \int_{Q(z,r)} |u_x|^3 < 4\varepsilon_0 = \varepsilon_*.$$

Thus Corollary 4.10 gives

$$\frac{1}{\varrho^5} \int_{Q(y,\varrho)} |u - u_{y,\varrho}|^3 \, \mathrm{d}z \leq C \varepsilon_* \left(\frac{\varrho}{r} \right)^{3\beta}$$

for every $y \in Q(z, r/2)$ and every $0 < \varrho \leq r/2$. Hölder continuity of u within $Q(z, r/2)$ now follows immediately from the Campanato Lemma (see Lemma 4.11). □

4.4.4 Partial regularity I: box-counting dimension

Blömker & Romito (2009) showed that if

$$\mathcal{T} := \{t \geq 0 : \ \|u\|_{H^1} \text{ is not essentially bounded in a neighbourhood of } t\}$$

then $d_B(\mathcal{T}) \leq 1/4$, where d_B denotes the box-counting dimension (see their Remark 4.7 – the proof is not actually given in their paper, but it follows easily from the estimates they obtain, using the argument from Robinson & Sadowski (2007)). Since $H^1(\mathbb{T}) \subset L^\infty(\mathbb{T})$, it follows in particular that if

$$\mathcal{T}_\infty := \{t \geq 0 : \ \|u\|_{L^\infty} \text{ is not essentially bounded in a neighbourhood of } t\}$$

then $\mathcal{T}_\infty \subseteq \mathcal{T}$, and so trivially $d_B(\mathcal{T}_\infty) \leq 1/4$. Since the set

$$S' := \{(x,t) \in \mathbb{T} \times [0,\infty) : \ u \text{ is not bounded in any neighbourhood of } (x,t)\}$$

(recall (1.29)) is a subset of $\mathcal{T}_\infty \times \mathbb{T}$, it follows from properties of the box-counting dimension that $d_B(S') \leq 5/4$.

We now use the conditional regularity of the previous section to prove a sharper estimate of the box-counting dimension of a larger set, that is the singular set defined in (1.29),

$$S = \{(x,t) \in \mathbb{T} \times [0,\infty) : \ u \text{ is not space-time Hölder continuous in any neighbourhood of } (x,t)\}.$$

We use the 'Minkowski definition' of the box-counting dimension in our argument, namely

$$d_B(K) := n - \liminf_{\delta \to 0^+} \frac{\log |K_\delta|}{\log \delta}, \qquad K \subset \mathbb{R}^n, \tag{4.46}$$

where $K_\delta := \{y\colon \mathrm{dist}(y,K) < \delta\}$ denotes the δ-neighbourhood of K. This formulation is one of a number of equivalent definitions of the box-counting dimension, see Proposition 2.4 in Falconer (2014).

Corollary 4.13 (Partial regularity I)**.** *For every compact $K \subset \mathbb{T} \times (0,T)$*

$$d_B(S \cap K) \leq 7/6.$$

The reason for considering the intersection $S \cap K$ (instead of S) is technical, recall the comments following (1.23). In fact, it suffices to take $S \cap (\mathbb{T} \times [a,b])$ (instead of $S \cap K$) for $0 < a < b$.

Proof. Let $\eta := \inf\{t^{1/4} \colon (x,t) \in K \text{ for some } x\}$. Given $r \in (0, \eta)$ let

$M_r :=$ maximal number of pairwise disjoint r-cylinders with centres in $S \cap K$,

$N_r :=$ minimal number of r-cylinders with centres in $S \cap K$ needed to cover $S \cap K$.

Step 1. We show that $M_r \leq cr^{-5/3}$ for sufficiently small r.

Let $Q(z_1, r), \ldots, Q(z_{M_r}, r)$ be a family of pairwise disjoint cylinders with centres $z_i \in S \cap K$ ($i = 1, \ldots, M_r$). Note that the choice of sufficiently small r above guarantees that these cylinders are contained within $\mathbb{T} \times (0, \infty)$. The conditional regularity result of Theorem 4.12 guarantees that for sufficiently small r

$$\frac{1}{r^2} \int_{Q(z_i,r)} |u_x|^3 \geq \varepsilon_0, \qquad i = 1, \ldots, M_r.$$

Thus, since Hölder's inequality gives

$$\int_{Q(z_i,r)} |u_x|^3 \leq c \left(\int_{Q(z_i,r)} |u_x|^{10/3} \right)^{9/10} r^{1/2},$$

we obtain, using (4.8),

$$\begin{aligned} c > \int_0^T \int |u_x|^{10/3} &\geq \sum_{i=1}^{M_r} \int_{Q(z_i,r)} |u_x|^{10/3} \\ &\geq c \sum_{i=1}^{M_r} \left(r^{-1/2} \int_{Q(z_i,r)} |u_x|^3 \right)^{10/9} \\ &\geq c \sum_{i=1}^{M_r} r^{5/3} \varepsilon_0^{10/9} \\ &= c M_r r^{5/3}. \end{aligned}$$

At this point it is interesting to note that since

$$d_B(S \cap K) \leq \limsup_{r \to 0} \frac{\log M_r}{-\log r}$$

this bound on M_r implies that $d_B(S \cap K) \leq 5/3$ (as in the context of the Navier–Stokes equations, see Robinson & Sadowski (2009)), but this does not improve on the bound 5/4 mentioned above. However, unlike in the case of the Navier–Stokes equations, the use of the Minkowski definition (4.46) gives a sharper bound (which is, in essence, a consequence of a dimensional analysis of the SGM; that is, roughly speaking, the dimension of time, 4, is larger than the space dimension, 1), which we show in the following steps.

Step 2. We show that $N_{2r} \le M_r$ for all $r \in (0, \eta/2)$.

Let $\{Q(z_i, r)\}_{i=1}^{M_r}$ be a family of pairwise disjoint cylinders with centres $z_i = (x_i, t_i) \in S \cap K$. We will show that the family $\{Q(z_i, 2r)\}_{i=1}^{M_r}$ covers $S \cap K$, which proves the inequality above. Indeed, suppose that this is not true, so that there exists $z_0 = (x_0, t_0) \in S \cap K$ such that

$$z_0 \notin \bigcup_{i=1}^{M_r} Q(z_i, 2r)$$

Then for each i

$$|x_0 - x_i| \ge 2r \qquad \text{or} \qquad |t_0 - t_i| \ge (2r)^4 > 2r^4,$$

which shows that

$$Q(z_0, r) \quad \text{and} \quad Q(z_i, r) \quad \text{are disjoint.}$$

Thus $\{Q(z_i, r)\}_{i=0}^{M_r}$ is a family of pairwise disjoint cylinders with centres in $S \cap K$, which contradicts the definition of M_r.

Step 3. We deduce that $d_B(S \cap K) \le 7/6$.

For $r < \min\{1, R_0, \eta/2\}$ let $\{Q(z_i, r)\}_{i=1}^{N_r}$ be a family of pairwise disjoint r-cylinders which cover $S \cap K$ with centres $z_i = (x_i, t_i) \in S \cap K$. Note that

$$(S \cap K)_{r^4} \subset \bigcup_{i=1}^{N_r} Q(z_i, 2r). \tag{4.47}$$

Indeed, given $z = (x, t) \in (S \cap K)_{r^4}$ let $z_0 \in S \cap K$ be such that $|z - z_0| < r^4$ and suppose that $z_0 = (x_0, t_0) \in Q(z_i, r)$ for some $i \in \{1, \ldots, N_r\}$. Then

$$|x - x_i| \le |x - x_0| + |x_0 - x_i| < r^4 + r < 2r,$$
$$|t - t_i| \le |t - t_0| + |t_0 - t_i| < 2r^4 < (2r)^4,$$

that is $z \in Q(z_i, 2r)$, which shows (4.47). Therefore, using steps 1 and 2, we obtain

$$|(S \cap K)_{r^4}| \le N_r 2^7 r^5 \le M_{r/2} 2^7 r^5 \le c\, r^{10/3}.$$

Letting $\delta := r^4$ we obtain

$$|(S \cap K)_\delta| \le c\, \delta^{5/6}$$

for all sufficiently small $\delta > 0$. Thus

$$\frac{\log|(S\cap K)_\delta|}{\log\delta} \geq \frac{\log c + \frac{5}{6}\log\delta}{\log\delta} \to \frac{5}{6} \qquad \text{as } \delta\to 0^+,$$

and so (4.46) gives $d_B(S\cap K)\leq 7/6$. □

Note that the above corollary gives in particular a similar bound on the Hausdorff dimension, $d_H(S\cap K)\leq 7/6$ (since $d_H(K)\leq d_B(K)$ for any compact K, by a property of the Hausdorff dimension, see, for example, Proposition 3.4 in Falconer (2014)), and so $|S\cap K|=0$ for any compact set K, which implies that $|S|=0$.

4.5 The second conditional and partial regularity results

Here we show that there exists $\varepsilon_1>0$ such that any suitable weak solution u is regular at $z=(x,t)$ whenever

$$\limsup_{r\to 0}\frac{1}{r}\int_{Q(z,r)} u_{xx}^2 \leq \varepsilon_1$$

or

$$\limsup_{r\to 0}\left\{ \operatorname{ess\,sup}_{s\in(t-r^4,t+r^4)}\frac{1}{r}\int_{B_r(x)} u(s)^2\right\} < \varepsilon_1.$$

Given $z=(x,t)$ we will write $B_r\coloneqq (x-r,x+r)$, $Q_r\coloneqq Q(z,r)$ and we will denote by $(u(s))_r\coloneqq (2r)^{-1}\int_{B_r}u(s)$ the mean of $u(s)$ over B_r. We will use the following quantities:

$$\begin{aligned}
A(r) &\coloneqq \operatorname{ess\,sup}_{s\in(t-r^4,t+r^4)}\frac{1}{r}\int_{B_r} u(s)^2\,\mathrm{d}x,\\
\overline{A}(r) &\coloneqq \operatorname{ess\,sup}_{s\in(t-r^4,t+r^4)}\frac{1}{r}\int_{B_r} \left(u(s)-(u(s))_r\right)^2\,\mathrm{d}x,\\
E(r) &\coloneqq \frac{1}{r}\int_{Q_r} u_{xx}^2\,\mathrm{d}z,\\
W(r) &\coloneqq \frac{1}{r^5}\int_{Q_r} |u|^3\,\mathrm{d}z,\\
\mathrm{Y}(r) &\coloneqq \frac{1}{r^2}\int_{Q_r} |u_x|^3\,\mathrm{d}z.
\end{aligned}$$

We note that each of the above quantities is invariant with respect to the scaling $u(x,t)\mapsto u(\lambda x,\lambda^4 t)$. Furthermore W and Y can be estimated in terms of A, $\overline{A}$ and E, which we make precise in the following lemma.

Lemma 4.14 (Interpolation inequalities)**.** *For every* $r>0$

$$\begin{aligned} W(r) &\leq cA(r)^{11/8}E(r)^{1/8}+cA(r)^{3/2}, &(4.48)\\ \mathrm{Y}(r) &\leq c\overline{A}(r)^{5/8}E(r)^{7/8}. &(4.49)\end{aligned}$$

Proof. Due to scale-invariance we can assume that $r=1$. As for the estimate on $W(1)$ we write $\overline{u}(t):=(u(t))_1$ and apply the decomposition

$$u(x,t)=(u(x,t)-\overline{u}(t))+\overline{u}(t)=:v(x,t)+\overline{u}(t)$$

Applying the $1D$ embedding $H^{1/6}\subset L^3$ and using the fact that $v(t)$ has zero mean we can write (for each t)

$$\|v\|^3_{L^3(B_1)}\leq c\|v\|^3_{H^{1/6}(B_1)}\leq c\|v\|^3_{\dot H^{1/6}(B_1)},$$

and so, by Sobolev interpolation,

$$\|v\|^3_{L^3(B_1)}\leq c\|v\|^{11/4}_{L^2(B_1)}\|\partial_{xx}v\|^{1/4}_{L^2(B_1)}\leq c\|u\|^{11/4}_{L^2(B_1)}\|\partial_{xx}u\|^{1/4}_{L^2(B_1)},$$

where we also used the fact that $\|v\|_{L^2(B_1)}\leq 2\|u\|_{L^2(B_1)}$. Thus

$$\begin{aligned}\int_{-1}^1\|v(t)\|^3_{L^3}\mathrm{d}t &\leq c\left(\operatorname{ess\,sup}_{t\in(-1,1)}\|v(t)\|_{L^2}\right)^{11/4}\left(\int_{-1}^1\|\partial_{xx}u(t)\|^{1/4}_{L^2}\mathrm{d}t\right)\\ &\leq c\,A(1)^{11/8}E(1)^{1/8}.\end{aligned}$$

We also have

$$\begin{aligned}\int_{-1}^1\|\overline{u}(t)\|^3_{L^3}\mathrm{d}t &= c\int_{-1}^1\left|\int_{-1}^1 u(x,t)\,\mathrm{d}x\right|^3\mathrm{d}t\leq c\int_{-1}^1\left(\int_{-1}^1 u(x,t)^2\mathrm{d}x\right)^{3/2}\mathrm{d}t\\ &\leq c\,A(1)^{3/2}.\end{aligned}$$

The last two inequalities show the required estimate on $W(1)$.

As for the estimate on $\mathrm{Y}(1)$, we let $v(x,t):=u(x,t)-(u(t))_1$ and write (for each t)

$$\begin{aligned}\|v_x\|^2_{L^3(B_1)} &\leq c\|v_x\|^2_{H^{1/6}(B_1)}\leq c\sum_{k\in\mathbb{Z}}\left(1+|k|^{1/3}\right)|\widehat{v_x}(k)|^2\\ &= c\sum_{k\neq 0}\left(k^2+|k|^{2+1/3}\right)|\widehat{v}(k)|^2\leq c\sum_{k\neq 0}|k|^{2+1/3}|\widehat{v}(k)|^2\\ &\leq c\|v\|^2_{\dot H^{7/6}(B_1)},\end{aligned}$$

where $\hat{f}(k)$ denotes the k-th Fourier mode in the Fourier expansion of f on $(-1,1)$. Applying Sobolev interpolation we obtain

$$\|v_x\|^3_{L^3(B_1)} \le c\|v\|^3_{\dot{H}^{7/6}(B_1)} \le c\|v\|^{5/4}_{L^2(B_1)}\|\partial_{xx}v\|^{7/4}_{L^2(B_1)},$$

and thus

$$\begin{aligned}\mathrm{Y}(1) &= \int_{-1}^{1}\|v_x(t)\|^3_{L^3}\mathrm{d}t \le c\left(\operatorname{ess\,sup}_{t\in(-1,1)}\|v(t)\|_{L^2}\right)^{5/4}\int_{-1}^{1}\|\partial_{xx}v(t)\|^{7/4}_{L^2}\mathrm{d}t\\ &\le c\,\overline{A}(1)^{5/8}E(1)^{7/8}.\end{aligned}$$

□

We can now state the main theorem of this section.

Theorem 4.15 (Conditional regularity in terms of u_{xx})**.** *Given $\beta\in(0,1)$ there exists an $\varepsilon_1>0$ such that if*

$$\limsup_{r\to 0}\frac{1}{r}\int_{Q(z,r)}u_{xx}^2<\varepsilon_1 \tag{4.50}$$

then u is β-Hölder continuous (as in (4.45)) in $Q(z,\rho)$ for some $\rho>0$.

Proof. The proof is inspired by Lin (1998) and Kukavica (2009*c*). Without loss of generality we can assume that $z=(0,0)$. We will show that (4.50) implies that

$$\mathrm{Y}(r)\le\varepsilon_0 \qquad \text{for some } r\in(0,R_0), \tag{4.51}$$

which, in the light of Theorem 4.12, proves the theorem.

Step 1. We show the estimate

$$\overline{A}(r/2)+E(r/2)\le\frac{1}{2}\overline{A}(r)+c\left(E(r)+E(r)^{10}\right) \qquad \text{for any } r>0.$$

Due to the scale invariance it is sufficient to take $r=1$. For brevity we will write $\overline{A}:=\overline{A}(1)$, $E:=E(1)$, $\mathrm{Y}:=\mathrm{Y}(1)$, as well as $B:=B_1$, $Q:=Q_1$. Let $\phi\in C_0^\infty(Q_{3/4};[0,1])$ be such that $\phi=1$ on $Q_{1/2}$ and $|\partial_t\phi|,|\partial_x^k\phi|\le c$ for all $k\le 4$.

Furthermore, let $\sigma\in C_0^\infty(B_1;[0,1])$ be such that $\sigma=1$ on $B_{3/4}$.

We set

$$u^\sigma(t):=\frac{\int_B u(t)\sigma\,\mathrm{d}x}{\int_B\sigma\,\mathrm{d}x} \qquad \text{and} \qquad [u]^\sigma:=\frac{\int_Q u\sigma}{\int_Q\sigma}.$$

In other words, recalling the notation (4.23), used in the proof of the Parabolic Poincaré inequality, we have $u^\sigma \equiv u_1^\sigma$, $[u]^\sigma \equiv [u]_1^\sigma$. Recall the Poincaré inequality (4.26),

$$\int_B |u(t) - u^\sigma(t)|^3 \, \sigma \le c \int_B |u_x(t)|^3 \sigma, \tag{4.52}$$

and Corollary 4.6,

$$\int_Q |u - [u]^\sigma|^3 \, \sigma \ \le c \left(Y + Y^2 \right). \tag{4.53}$$

Observe also that for almost every $t \in (-1,1)$

$$|u^\sigma(t) - [u]^\sigma|^3 \le c \left(Y + Y^2 \right), \tag{4.54}$$

due to (4.29).

The local energy inequality (4.15) for $u - [u]^\sigma$ (recall the comments following (4.15)) gives

$$\begin{aligned}
\overline{A}(1/2) + E(1/2) &\le 2 \operatorname{ess\,sup}_{s\in(-2^{-4},2^{-4})} \int_{B_{1/2}} (u(s) - [u]^\sigma)^2 + 2 \int_{Q_{1/2}} u_{xx}^2 \\
&\le c \int_{Q_{3/4}} (u - [u]^\sigma)^2 + c \int_Q \left(u_x^2 + |u_x|^3 \right) + c \int_{-(3/4)^4}^{(3/4)^4} \left| \int_{B_{3/4}} u_x^2 \left(u - [u]^\sigma \right) \phi_{xx} \right| \\
&\le c \left(\int_{Q_{3/4}} |u - [u]^\sigma|^3 \right)^{2/3} + c(\mathrm{Y}^{2/3} + \mathrm{Y}) + c \int_{Q_{3/4}} u_x^2 \left| \left(u - u^\sigma \right) \phi_{xx} \right| \\
&\qquad + c \int_{-1}^1 \left| \left(u^\sigma - [u]^\sigma \right) \int_B u_x^2 \phi_{xx} \right|,
\end{aligned}$$

where we used the fact that $\int_B (f - (f)_1)^2 \le \int_B (f - K)^2$ for any $K \in \mathbb{R}$, $f \in L^2(B)$ in the first line, the fact that $\operatorname{supp} \phi \subset Q_{3/4}$ in the second line, and Hölder's inequality and triangle inequality in the third line. Now, by applying (4.53) to the first of the resulting terms and integrating the last term by parts, we obtain

$$\begin{aligned}
\overline{A}(1/2) + E(1/2) &\le c(\mathrm{Y}^{2/3} + \mathrm{Y}^{4/3}) + c \int_{Q_{3/4}} u_x^2 \left| u - u^\sigma \right| \\
&\qquad + c \int_{-1}^1 \left| \left(u^\sigma - [u]^\sigma \right) \int_B u_x u_{xx} \phi_x \right| \\
&\le c(\mathrm{Y}^{2/3} + \mathrm{Y}^{4/3}) + c\mathrm{Y}^{2/3} \left(\int_Q |u - u^\sigma|^3 \, \sigma \right)^{1/3} \\
&\qquad + c \left(\operatorname{ess\,sup}_{s\in(-1,1)} |u^\sigma(s) - [u]^\sigma| \right) \int_Q |u_x u_{xx}|,
\end{aligned}$$

where we also applied Hölder's inequality and used the fact that $\sigma = 1$ on $Q_{3/4}$ in the second line. Finally, applying (4.52), (4.54), the Cauchy-Schwarz inequality and Hölder's inequality gives

$$\begin{aligned}\overline{A}(1/2) + E(1/2) &\le c(\mathrm{Y}^{2/3} + \mathrm{Y}^{4/3}) + c\left(\mathrm{Y}^{1/3} + \mathrm{Y}^{2/3}\right)\mathrm{Y}^{2/3}E^{1/2}\\ &= c(\mathrm{Y}^{2/3} + \mathrm{Y}^{4/3}) + cE^{1/2}\left(\mathrm{Y} + \mathrm{Y}^{4/3}\right)\\ &\le c\left(\overline{A}^{5/12}E^{7/12} + \overline{A}^{5/6}E^{7/6} + \overline{A}^{5/8}E^{11/8} + \overline{A}^{5/6}E^{5/3}\right)\\ &\le \frac{1}{2}\overline{A} + c\left(E + E^{10}\right),\end{aligned}$$

as required, where we also used the interpolation inequality (4.49) in the third line, and Young's inequality $ab \le \delta a^p + C_\delta b^q$ (where $1/p + 1/q = 1$ and sufficiently small $\delta > 0$) in the last line.

Step 3. We show (4.51).

Let $\varepsilon_1 > 0$ be small enough that

$$c\left(\varepsilon_1 + \varepsilon_1^{10}\right) \le \frac{1}{4}\varepsilon_0^{2/3}.$$

By assumption there exists r_0 such that $E(r) < \varepsilon_1$ for $r \in (0, r_0]$. From Step 2

$$\overline{A}(r/2) + E(r/2) \le \frac{1}{2}\overline{A}(r) + \frac{1}{4}\varepsilon_0^{2/3}, \quad r \in (0, r_0],$$

and iterating this inequality k times we obtain

$$\overline{A}(2^{-k}r_0) + E(2^{-k}r_0) \le 2^{-k}\overline{A}(r_0) + \frac{1}{4}\varepsilon_0^{2/3}\sum_{j=0}^{k-1} 2^{-j} \le 2^{-k}\overline{A}(r_0) + \frac{1}{2}\varepsilon_0^{2/3}.$$

Thus for sufficiently large k

$$\overline{A}(2^{-k}r_0) + E(2^{-k}r_0) \le \varepsilon_0^{2/3},$$

and so interpolation inequality (4.49) gives

$$\mathrm{Y}(2^{-k}r_0) \le \overline{A}(2^{-k}r_0)^{5/8}E(2^{-k}r_0)^{7/8} \le \varepsilon_0^{5/12}\varepsilon_0^{7/12} = \varepsilon_0,$$

as required. □

Corollary 4.16 (Conditional regularity in terms of $\operatorname{ess\,sup}_t \int_{B_r} u(t)^2$). *There exists an $\varepsilon_2 > 0$ such that if*

$$\limsup_{r\to 0} \left\{ \operatorname{ess\,sup}_{s\in(t-r^4,t+r^4)} \frac{1}{r} \int_{B_r(x)} u(s)^2 \right\} < \varepsilon_2 \tag{4.55}$$

then u is β-Hölder continuous in $Q(z,\rho)$ for some $\rho > 0$.

Proof. The claim follows by replacing the estimate from Step 1 above by

$$A(r/2) + E(r/2) \le \frac{1}{2} E(r) + c\left(A(r) + A(r)^5\right), \tag{4.56}$$

whose proof we defer for a moment. Indeed, then one can choose $\varepsilon_2 > 0$ sufficiently small such that $c\,(\varepsilon_2 + \varepsilon_2^5) \le \varepsilon_0^{2/3}/4$ and the claim follows as in Step 3 above by noting that $\overline{A} \le A$. We now verify (4.56), where we assume that $r = 1$, as before. Using the local energy inequality (4.15) we obtain

$$\begin{aligned} A(1/2) + E(1/2) &\le c \int_Q \left(u^2 + u_x^2 + |u_x|^3 + u_x^2|u|\right) \\ &\le c\left(A + \mathrm{Y}^{2/3} + \mathrm{Y} + \mathrm{Y}^{2/3} W^{1/3}\right) \\ &\le c\left(A + A^{5/12}E^{7/12} + A^{5/8}E^{7/8} + A^{43/24}E^{17/24} + A^{23/12}E^{7/12}\right) \\ &\le \frac{1}{2}E + c\left(A + A^5\right), \end{aligned}$$

as required, where we used Hölder's inequality in the second line, the interpolation inequalities (4.48), (4.49) (together with a fact that $\overline{A} \le A$) in the third line, and Young's inequality $ab \le \delta a^p + c_\delta b^q$ (where $1/p + 1/q = 1$ and $\delta > 0$ is chosen sufficiently small). □

Using Theorem 4.15 we can obtain improved bounds on the dimension of the singular set in terms of the (parabolic) Hausdorff measure. For a set $X \subset \mathbb{R} \times \mathbb{R}$ and $k \ge 0$ let

$$P^k(X) \coloneqq \lim_{\delta\to 0^+} P^k_\delta(X) \tag{4.57}$$

denote the k-dimensional parabolic Hausdorff measure, where

$$P^k_\delta(X) \coloneqq \inf\left\{ \sum_{i=1}^\infty r_i^k \,:\, X \subset \bigcup_i Q_{r_i} \,:\, r_i < \delta \right\},$$

and $Q_{r_i} = Q_{r_i}(x,t)$ is a r_i-cylinder, $i \ge 1$. Observe that $P^1(X) = 0$ if and only if for every $\delta > 0$ the set X can be covered by a collection $\{Q_{r_i}\}$ such that $\sum_i r_i < \delta$.

Corollary 4.17 (Partial regularity II). *The singular set S of a suitable weak solution of* (4.1) *satisfies* $\mathcal{P}^1(S)=0$.

Note that this in particular gives $d_H(S)\leq 1$ (since $\mathcal{H}^1(S)\leq c\mathcal{P}^1(S)$, where $\mathcal{H}^1$ denotes the 1-dimensional Hausdorff measure).

We will need the Vitali Covering Lemma in the following form: given a family of parabolic cylinders $Q_r(x,t)$, there exists a countable (or finite) disjoint subfamily $\{Q_{r_i}(x_i,t_i)\}$ such that for any cylinder $Q_r(x,t)$ in the original family there exists an i such that $Q_r(x,t)\subset Q_{5r_i}(x_i,t_i)$. (For a proof see Caffarelli, Kohn & Nirenberg (1982).)

Proof. Fix $\delta>0$ and let V be an open set containing S such that

$$\frac{5}{\varepsilon_1}\int_V u_{xx}^2\leq\delta.$$

Such V exists since $u_{xx}\in L^2(\mathbb{T}\times(0,T))$ (recall (4.5)) and since $|S|=0$ (see the comments preceding this section). For each $(x,t)\in S$, choose $r\in(0,\delta)$ such that $Q_{r/5}(x,t)\subset V$ and

$$\frac{5}{r}\int_{Q_{r/5}(x,t)} u_{xx}^2>\varepsilon_1.$$

Such a choice is possible, for otherwise the point (x,t) would be regular due to Theorem 4.15. We now use the Vitali Covering Lemma to extract a countable (or finite) disjoint subcollection of these cylinders $\{Q_{r_i/5}(x_i,t_i)\}$ such that the singular set S is still covered by $\{Q_{r_i}(x_i,t_i)\}$. Then

$$\sum_i r_i\leq\frac{5}{\varepsilon_1}\sum_i\int_{Q_{r_i/5}(x_i,t_i)} u_{xx}^2\leq\frac{5}{\varepsilon_1}\int_V u_{xx}^2\leq\delta,$$

as required. □

4.6 Remarks

We have proved two conditional regularity results, which imply two bounds on singular space-time set for the surface growth model:

$$d_B(S\cap K)\leq 7/6 \qquad \text{and} \qquad P^1(S)=0$$

for any compact $K\subset\mathbb{T}\times(0,\infty)$, where

$$S=\{(x,t)\in\mathbb{T}\times[0,\infty):\ u \text{ is not space-time Hölder continuous in any neighbourhood of } (x,t)\}.$$

Recall that it is not yet clear that local Hölder continuity (in space-time) of a suitable weak solution to the SGM implies smoothness in space-time.

In the next chapter we provide another local regularity criterion, which provides a sufficient condition for C^∞ smoothness.

Chapter 5

A sufficient integral condition for local regularity for the surface growth model

In this chapter we focus on the surface growth model,

$$u_t + u_{xxxx} + \partial_{xx} u_x^2 = 0 \qquad \text{on the torus } \mathbb{T}.$$

We will be interested in studying the behaviour of weak solutions to this model locally in space-time.

Namely we will study a property that is similar to the so-called local Serrin condition in the case of the Navier–Stokes equations. In the case of the Navier–Stokes equations this condition reads: if u is a weak solution on a space-time domain $U \times (t_1, t_2)$ such that

$$u \in L^{q'}((t_1, t_2); L^q(U)) \quad \text{ with } \frac{2}{q'} + \frac{3}{q} \leq 1 \tag{5.1}$$

then u is smooth in the space variables on this domain (recall (1.32) in the introduction). The condition is named after Serrin (1962 & 1963), who was the first to study the property (5.1) in the subcritical case (that is when the inequality in (5.1) is sharp "$<$"). The critical case (that is when $2/q' + 3/q = 1$) has been studied by Fabes et al. (1972) when $q \in (3, \infty)$, and Struwe (1988), Takahashi (1990) when $q > 3$. The most difficult case of $q' = \infty, q = 3$ was resolved by Escauriaza, Seregin & Šverák (2003). We refer the reader to an excellent presentation of the local Serrin condition in the Navier–Stokes equations and further references in Chapter 13 of Robinson et al. (2016).

Here, we prove that a weak solution u to the surface growth model on a space-time domain $U \times (t_1, t_2)$ is smooth on this domain provided that (1.31) holds, that is

$$u_x \in L^{q'}((t_1,t_2); L^q(U)) \quad \text{where } \frac{4}{q'} + \frac{1}{q} \le 1 \text{ and } q' < \infty \tag{5.2}$$

(and $q, q' \ge 2$), see Theorem 5.6 (which also covers the case $1/q + 4/q' < 1$, $q' = \infty$) and Theorem 5.7. Remarkably, under this condition we obtain smoothness in both space and time (rather than smoothness in space only, as in the case of the Navier–Stokes equations), which seems to be a consequence of the lack of the pressure function in the model. Note that we state the Serrin condition (5.2) in terms of u_x (rather than in terms of u, which is the case in the Navier–Stokes equations). This is related to the fact that the lowest spatial derivative of u involved in the nonlinear term in the surface growth model (4.4) is u_x (rather than the 0-th derivative, which is the case in the Navier–Stokes equations). Note that the condition from the partial regularity theory is also stated in terms of u_x (recall the first claim of Theorem 1.9). Therefore, the main result of this chapter is a next step towards understanding the remarkable similarity between the surface growth model and the Navier–Stokes equations.

Interestingly, local Hölder continuity of u follows trivially if the subcritical Serrin condition is satisfied (that is when (5.2) holds with $4/q' + 1/q < 1$) by a use of an extended version of the parabolic Poincaré inequality. Indeed the parabolic Poincaré inequality (4.21) can be extended to

$$\frac{1}{r^5} \int_{Q(z_0,r/2)} |u - u_{z_0,r/2}|^p \le C \left((r^\varepsilon M)^p + (r^\varepsilon M)^{2p} \right), \tag{5.3}$$

where

$$M := \left[\int_{t_1}^{t_2} \left(\int_U |u_x(t)|^p \right)^{p'/p} \mathrm{d}t \right]^{1/p'}$$

and $p, p' \in [2, \infty)$ are such that $1/p + 4/p' = 1 - \varepsilon$. Such an extension can be proved in the same way as (4.21) (recall Theorem 4.5). Thus if the condition (5.2) is satisfied with $1/q + 4/q' = 1 - \varepsilon$, then an application of Hölder's inequality to (5.3) together with the Campanato lemma (Lemma 4.11) give ε-Hölder continuity of u in $U \times (t_1, t_2)$.

Recall also the argument from the introduction (Section 1.4) which shows that the partial regularity theory gives α-Hölder continuity for any $\alpha \in (0,1)$ if the condition (5.2) is satisfied with $q, q' \ge 3$, $q' < \infty$. Therefore, since the main result of this chapter gives C^∞ smoothness (rather than merely α-Hölder continuity for any $\alpha \in (0,1)$), it also suggests that the definition of the singular set (1.29) is optimal.

The proof of our result begins with the approach of Takahashi (1990), adapted to the one-dimensional fourth-order setting. However, in order to show boundedness of higher derivatives in space, we use fractional order Sobolev spaces to obtain, roughly speaking, a half of a derivative at a time (see step 2' of the proof of Theorem 5.6). This seems to be a novel approach in this context. Moreover, our setting requires a uniqueness theorem for weak solutions of the biharmonic heat equation $w_t + w_{xxxx} = 0$. Since we are dealing with a fourth order equation, we cannot adapt any uniqueness theorem for the heat equation that uses the maximum principle. We can, however, adapt a uniqueness theorem for the heat equation that is based on the density of the image of the heat operator $\partial_t - \Delta$ in L^p. A uniqueness theorem of this kind is presented in Section 4.4.2 of Giga et al. (2010) and we prove an appropriate adaptation of it in Theorem 5.4.

The results of this chapter have been posted on the ArXiv (see Ożański (2018)) and have been submitted for publication.

The structure of this chapter is the following. In Section 2.2 we discuss the preliminary concepts including fractional Sobolev spaces (Section 5.1.1), the biharmonic heat kernel (Section 5.1.2), the uniqueness theorem (Section 5.1.3) and the concept of local weak solutions to the surface growth model (Section 5.1.4). We then proceed to the proof of the local Serrin condition for the surface growth model in Section 5.2. There, we discuss a certain representation formula for u_x and we prove regularity, first for the subcritical case $1/q + 4/q' < 1$ in Section 5.2.1 and then for the critical case $1/q + 4/q' = 1$ with $q' < \infty$ in Section 5.2.2.

5.1 Preliminaries

In the following we will say that a function satisfies a partial differential equation in an open set if it satisfies the distributional form of the equation. We fix $T > 0$. We denote a space-time cylinder by Q, that is $Q = B \times I$ for some intervals $B \subset \mathbb{R}$, $I \subset (0,T)$. We use the shorthand notation $\| \cdot \|_p := \| \cdot \|_{L^p(\mathbb{R})}$. Given $Q = B \times I$, where $B, I \subset \mathbb{R}$, $p, p' \in [1, \infty]$ we let

$$L^{p',p}(Q) = L^{p'}(I; L^p(B)) := \{ f \in Q \to \mathbb{R} \, : \, f \text{ is measurable and } \|f\|_{L^{p',p}(Q)} < \infty \},$$

where

$$\|f\|^{p'}_{L^{p',p}(Q)} := \int_I \|f(t)\|^{p'}_{L^p(B)} \mathrm{d}t$$

for $p' < \infty$ and

$$\|f\|_{L^{\infty,p}(Q)} := \operatorname{esssup}_{t\in I}\|f(t)\|_{L^p(B)}.$$

We use the shorthand notation $L^{p',p} := L^{p',p}(\mathbb{R}\times(0,T))$ and $\|\cdot\|_{p',p} := \|\cdot\|_{L^{p',p}(\mathbb{R}\times(0,T))}$. This should not be confused with the weak-L^p spaces, which we do not use in this chapter, except for a brief encounter in the proof of Theorem 5.1 below. This also should not be confused with some literature on the Navier–Stokes equations where the order of the indices p', p is switched; for example, the $L^{3,\infty}$ condition from Escauriaza et al. (2003) corresponds to $L^{\infty,3}$ in our notation.

5.1.1 Fractional Sobolev spaces

We denote by $\widehat{f}$ the Fourier transform (in the x variable) of f, that is

$$\widehat{f}(\xi) := \int_{\mathbb{R}} f(x)\mathrm{e}^{-2\pi i x\xi}\mathrm{d}x, \qquad \text{for } \xi\in\mathbb{R}.$$

We will only consider Fourier transforms of functions that are bounded and have compact support. For such functions and any $s>0$ we denote by $\Lambda^s f$ the function with Fourier transform

$$\widehat{\Lambda^s f} := |\xi|^s\widehat{f}(\xi).$$

Let

$$H^s := \{f\in L^2(\mathbb{R}) \,:\, \Lambda^s f\in L^2(\mathbb{R})\}$$

denote the fractional Sobolev space of order s with the norm defined by

$$\|f\|_{H^s}^2 := \int_{\mathbb{R}} \left(1+|\xi|^{2s}\right)|\widehat{f}(\xi)|^2\mathrm{d}\xi.$$

Observe that $\Lambda f\in L^2$ if and only if $f_x\in L^2$ with

$$2\pi\|\Lambda f\|_2 = \|f_x\|_2. \tag{5.4}$$

Recall the Sobolev–Slobodeckij characterisation $H^s = W^{s,2}(\mathbb{R})$ for $s\in(0,1)$ with

$$\|f\|_{H^s} \simeq \|f\|_{W^{s,2}(\mathbb{R})}, \tag{5.5}$$

where "$\simeq$" denotes the equivalence of norms, $W^{s,2}(\Omega) := \{f \,:\, \|f\|_{W^{s,2}(\Omega)}<\infty\}$ and

$$\|f\|_{W^{s,2}(\Omega)}^2 := \left(\int_\Omega |f(x)|^2\mathrm{d}x + \int_\Omega\int_\Omega \frac{|f(x)-f(y)|^2}{|x-y|^{1+2s}}\mathrm{d}y\,\mathrm{d}x\right) \tag{5.6}$$

for any open $\Omega\subset\mathbb{R}$. (For the proof of this characterisation see, for example, Proposition 3.4 in Di Nezza et al. (2012).)

5.1.2 Biharmonic heat kernel

Let

$$\Phi(x,t) := \frac{c}{t^{1/4}} K(|x|/t^{1/4}) \tag{5.7}$$

where

$$K(r) := \int_0^\infty \mathrm{e}^{-s^4} \cos(r\, s)\, \mathrm{d}s$$

and $c > 0$ is such that $\|\Phi(t)\|_1 = 1$. Then Φ is the biharmonic heat kernel (that is $w := \Phi(t) * f$ satisfies the biharmonic heat equation $w_t + w_{xxxx} = 0$ with initial condition $w(0) = f$, see Ferrero et al. (2008)). Note that, since $K(|x|)$ is smooth in $x \in \mathbb{R}$ and has exponential decay as $|x| \to \infty$, we obtain

$$\|\partial_x^{(k)} \Phi(t)\|_p \leq C t^{-(k+1-1/p)/4}, \qquad k \geq 0. \tag{5.8}$$

Moreover

$$\left| |\xi|^s \widehat{\Phi}(\xi,t) \right| \leq C t^{-s/4}, \qquad s > 0, t \in (0,T), \tag{5.9}$$

for $\xi \in \mathbb{R}$, $t > 0$, $s \geq 0$, where $\widehat{\Phi}$ denotes the Fourier transform (with respect to x) of Φ. The estimate (5.9) (as well as (5.8)) follows directly from the formula (5.7) and from the smoothness of K. The estimate in (5.8) gives the following.

Theorem 5.1 (Estimates for the convolution with the biharmonic heat equation)**.** *If $f \in L^1_{\mathrm{loc}}(\mathbb{R} \times [0,T))$, $0 \leq k \leq 3$ and*

$$v(t) := \int_0^t \Phi(t-s) \partial_x^{(k)} f(s)\, \mathrm{d}s$$

then

$$\|v\|_{r',r} \leq C_{l,l',r,r'} \|f\|_{l',l},$$

l, l', r, r' satisfy $1 \leq l \leq r \leq \infty$, $1 \leq l' \leq r' \leq \infty$ and either

$$\frac{1}{l} + \frac{4}{l'} < \frac{1}{r} + \frac{4}{r'} + (4-k) \tag{5.10}$$

or

$$\frac{1}{l} + \frac{4}{l'} \leq \frac{1}{r} + \frac{4}{r'} + (4-k) \quad \textit{and } 1 < l' < r' < \infty. \tag{5.11}$$

Note that the cases of $l', r' \in \{1, \infty\}$ and $l' = r'$ are allowed in (5.10), but not in (5.11).

Proof. We first focus on the case (5.10). We have

$$v(t)=\int_0^t \Phi(t-s)*\partial_x^{(k)}f(s)\,\mathrm{d}s=(-1)^k\int_0^t \partial_x^{(k)}\Phi(t-s)*f(s)\,\mathrm{d}s.$$

Thus, Young's inequality gives

$$\|v(t)\|_r\le\int_0^t\|\partial_x^{(k)}\Phi(t-s)\|_a\|f(s)\|_l\,\mathrm{d}s\le C\int_0^t(t-s)^{-(k+1-1/a)/4}\|f(s)\|_l\,\mathrm{d}s,$$

where $1/r=1/a+1/l-1$. Hence

$$\|v(t)\|_{r',r}\le C\|f\|_{l',l}\|\,|s|^{-(k+1-1/a)/4}\|_{a'}, \tag{5.12}$$

where $1/r'=1/a'+1/l'-1$. The last norm is finite if and only if $a'(k+1-1/a)/4<1$, which is equivalent to (5.10).

As for the case (5.11), we need to use the Young inequality for weak spaces,

$$\|f*g\|_{r'}\le\|f\|_{l'}\|g\|_{\widetilde{L}^{a'}}, \tag{5.13}$$

where $r',l',a'\in(1,\infty)$ are such that $1/r'=1/a'+1/l'-1$,

$$\|g\|_{\widetilde{L}^{a'}}\coloneqq\inf\{C>0\,:\,d_g(\alpha)\le C^{a'}/\alpha^{a'}\quad\text{for all }\alpha>0\}$$

denotes the norm of the weak-$L^{a'}$ space, and

$$d_g(\alpha)\coloneqq|\{|g|>\alpha\}|,\qquad\text{for }\alpha>0$$

denotes the distribution function of g. We refer the reader to Theorem 11.3 in McCormick et al. (2013) for a proof of (5.13).

The point of using the weak form of Young's inequality (5.13) is that $t^{-1/a'}$ is an element of the weak-$L^{a'}(0,1)$ space (but not of $L^{a'}(0,1)$), and so using it in the step leading to (5.12) gives

$$\|v(t)\|_{r',r}\le C\|f\|_{l',l}\|\,|s|^{-(k+1-1/a)/4}\|_{\widetilde{L}^{a'}},$$

where $1/r'=1/a'+1/l'-1$ (note $a',r',l'\in(1,\infty)$ by (5.11)). Thus the claim follows since the last norm is finite if and only if $a'(k+1-1/a)/4\le l$, which is guaranteed by (5.11). □

We will often consider a function $v:\mathbb{R}\times[0,T)\to\mathbb{R}$ of the form

$$v(t)=\int_0^t\Phi(t-s)*g(s)\,\mathrm{d}s, \tag{5.14}$$

where

$$g = \sum_{k=0}^{3} \phi_k \partial_x^k f_k, \tag{5.15}$$

$f_k \in L^{l'} L^l$ (for some $l', l \geq 1$), $\phi_k \in C_0^\infty(Q)$ for some fixed $Q \Subset \mathbb{R} \times (0,T)$. Since (5.14) is not necessarily well-defined for such f_k's (i.e. their derivatives might not exist), we will understand (5.14) as if all derivatives are transferred onto Φ and ϕ_k's (via integration by parts). Namely (5.14) means that

$$v(t) = \sum_k \sum_{j=0}^{k} (-1)^k \binom{k}{j} \int_0^t \partial_x^{(j)} \Phi(t-s) * \left[f_k(s) \partial_x^{(k-j)} \phi_k(s) \right] \mathrm{d}s.$$

We now formulate a corollary of Theorem 5.1 which is "tailor-made" for v's of such form.

Corollary 5.2. *Let v be given by* (5.14) *with $f_k \in L^{l'_k, l_k}(Q)$, where l'_k, l_k satisfy* (5.10) *or* (5.11) *for some r, r'. Then*

$$\|v\|_{r',r} \leq \sum_k C_k \|f_k\|_{L^{l'_k, l_k}(Q)}.$$

Corollary 5.3 (Representation formula for (5.14))**.** *Suppose that $r', r \in [1, \infty]$ and $w \in L^{r',r}$ is a distributional solution of*

$$w_t + w_{xxxx} = g \quad \text{in } \mathbb{R} \times (0,T)$$

where $k \leq 3$, g is given by (5.15) *and each f_k belongs to $L^{l'_k, l_k}(Q)$ with l'_k, l_k satisfying* (5.10) *or* (5.11)*. Suppose further that $w = 0$ in $\mathbb{R} \times (0, t_0)$ for some $t_0 > 0$. Then w is given by the convolution with the biharmonic heat kernel, namely it satisfies the representation* (5.14)*.*

Proof. Let

$$\widetilde{w}(t) := \int_0^t \Phi(t-s) * g(s) \, \mathrm{d}s.$$

Since $k \leq 3$ and $f_k \in L^{l'_k} L^{l_k}(Q)$, Corollary 5.2 gives $\widetilde{w} \in L^{r',r}$. Moreover $\widetilde{w}$ satisfies $\widetilde{w}_t + \widetilde{w}_{xxxx} = g$, similarly as w. Thus $\widetilde{w} = w$ due to the uniqueness of solutions to the biharmonic heat equation, see the theorem below. □

5.1.3 Uniqueness of solutions to homogeneous biharmonic heat equation

Theorem 5.4 (Uniqueness of solutions to homogeneous biharmonic heat equation)**.** *Suppose that $q, q' \in [1,\infty]$ and $v \in L^{q',q}$ is a distributional solution to the homogeneous biharmonic heat equation, that is*

$$\int_0^T \int_{\mathbb{R}} v(\phi_t - \phi_{xxxx}) = 0 \qquad \text{for all } \phi \in C_0^\infty(\mathbb{R}\times[0,T)). \tag{5.16}$$

Then $v = 0$.

Proof. We modify the argument from Section 4.4.2 of Giga et al. (2010). We focus on the case $T<\infty$ (the case $T=\infty$ follows trivially by applying the result for all $T>0$).

We first observe that the assumption $v \in L^{q',q}$ implies that (5.16) holds also for all $\phi \in C^\infty(\mathbb{R}\times[0,T))$ such that

$$\begin{cases} \partial_t^k \partial_x^m \phi \in L^{p',p} & \text{for every } k,m \geq 0, \\ \operatorname{supp}\phi \subset \mathbb{R}\times[0,T') & \text{for some } T'<T, \end{cases} \tag{5.17}$$

where $1/p+1/q=1$, $1/p'+1/q'=1$. Indeed, given such ϕ one can consider

$$\phi_j(x,t) \coloneqq \theta_j(x)\phi(x,t),$$

where $\theta_j(x) \coloneqq \theta(x/j)$ and $\theta \in C^\infty(\mathbb{R};[0,1])$ is any function such that $\theta(\tau)=1$ for $|\tau|\leq 1$ and $\theta(\tau)=0$ for $\tau \geq 2$. Then ϕ_j can be used as a test function in (5.16) and a simple use of the Dominated Convergence Theorem together with the properties (5.17) and the assumption $v\in L^{q',q}$ proves the claim.

We will show that

$$\int_0^T \int v\Psi = 0$$

for all $\Psi \in C_0^\infty(\mathbb{R}\times(0,T))$. The claim of the theorem then follows from the fundamental lemma of calculus of variations.

Given $\Psi \in C_0^\infty(\mathbb{R}^3\times(0,T))$ let $T'<T$ be such that $\Psi(t)\equiv 0$ for $t\geq T'$. Extend Ψ by zero for $t\leq 0$ and $t \geq T$. Let

$$\phi(x,t) \coloneqq -\int_0^{T-t} \Phi(T-t-s)\Psi(T-s)\,\mathrm{d}s,$$

where Φ denotes the biharmonic heat kernel (see (5.7)). Then ϕ solves the biharmonic heat equation backwards from T with right-hand side Ψ, that is

$$\begin{cases} \phi_t - \phi_{xxxx} = \Psi & \text{in } \mathbb{R}\times(-\infty,T), \\ \phi(T)=0. \end{cases} \tag{5.18}$$

In fact, we have $\phi(t) = 0$ for $t \in [T', T]$. By the biharmonic heat estimates (see (5.8)) we see that ϕ satisfies (5.17). Thus (5.16) gives

$$0 = \int_0^T \int v(\phi_t - \phi_{xxxx}) = \int_0^T \int v\Psi,$$

as required. □

5.1.4 Weak solutions of the surface growth model

Here we define the notion of a weak solution to the surface growth model. Since the local Serrin condition is concerned with behaviour of weak solutions in bounded cylinders in space-time, we focus only on the notion of a local solution (in space-time; rather than a solution to the initial value problem).

Definition 5.5 (Weak solution of the SGM)**.** *We say that $u \in L^{\infty,2}(Q)$ is a* weak solution *of the surface growth model on a cylinder Q if $u_{xx} \in L^{2,2}(Q)$ and*

$$\int_Q \left(u\, \phi_t - u_{xx}\phi_{xx} - u_x^2 \phi_{xx} \right) = 0$$

for all $\phi \in C_0^\infty(Q)$.

In what follows we will assume that u is a weak solution of the SGM on a given Q. Note that any weak solution u on a cylinder Q satisfies

$$u_x \in L^{10/3,10/3}(Q) \tag{5.19}$$

(which can be shown using Sobolev interpolation; see (2.5) in Ożański & Robinson (2017) for details), and so in particular the integral in the equation above is well-defined. Moreover

$$u_x \in L^{16/3,2}(Q), \tag{5.20}$$

given the main assumption of this chapter (i.e. (5.2)) is satisfied with $q \in (1, 2]$, which follows by a simple application of Lebesgue interpolation (in space and then in time) between (5.19) and (5.2). We will need (5.20) in order to circumvent a certain technical issue in the proof of the main result when $q \in (1, 2]$ (see the comments following (5.27) for details).

5.2 Proof of the main result

Here we show that if $Q \Subset \mathbb{R} \times (0,T)$ and $u_x \in L^{q',q}(Q)$ for any $q, q' \in (1,\infty]$ such that either

$$\frac{1}{q} + \frac{4}{q'} < 1 \quad \text{or} \quad \frac{1}{q} + \frac{4}{q'} = 1 \tag{5.21}$$

then $u \in C^\infty(Q)$.

The main idea of the proof is to note that the function

$$v := u_x$$

satisfies the equation

$$v_t + v_{xxxx} = -\partial_{xxx} v^2$$

in Q, and that one can apply the estimate from Theorem 5.1 to increase the regularity of v. In order to apply this strategy, we need to extend v to $\mathbb{R} \times (0,T)$ and explore the representation of such an extension by a formula similar to (5.14). To be precise, given a cutoff function $\phi \in C_0^\infty(Q; [0,1])$, let

$$w := v\phi.$$

Then w satisfies

$$w_t + w_{xxxx} = -(wv)_{xxx} + f_v \qquad \text{in } \mathbb{R} \times (0,T), \tag{5.22}$$

where

$$f_v := (v\,\phi_t + 4v_{xxx}\phi_x + 6v_{xx}\phi_{xx} + 4v_x\phi_{xxx} + v\,\phi_{xxxx}) + (3\phi_x\partial_{xx}v^2 + 3\phi_{xx}\partial_x v^2 + \phi_{xxx}v^2),$$

which we will write more concisely as

$$f_v = \sum_{m=0}^{3} \phi_m \partial_x^m v + \sum_{k=0}^{2} \psi_k \partial_x^k (v^2) \tag{5.23}$$

for some $\psi_k, \phi_m \in C_0^\infty(Q)$ (each being a constant multiple of a derivative of ϕ).

We note that w satisfies the representation formula

$$w(t) = \int_0^t \Phi_{xxx}(t-s) * [w(s)v(s)]\,\mathrm{d}s + \int_0^t \Phi(t-s) * f_v(s)\,\mathrm{d}s \quad \text{for } t \in (0,T), \tag{5.24}$$

where the last term is understood in the same sense as (5.14). Indeed, since $v \in L^{10/3}(Q)$ (recall (5.19)) we see that $wv, v^2 \in L^{5/3}(Q)$ and $w \in L^{10/3}(Q) \subset L^{5/3}(Q)$ (in particular $w \in L^{5/3,5/3}$ as $w = 0$ outside Q), and so we can use Corollary 5.3 (since the choice $r = r' = 5/3$, $l = l' = 5/3$ satisfies (5.10) trivially for any $k \in \{0,1,2,3\}$) to obtain (5.24).

5.2.1 The subcritical case

Here we focus on the subcritical case, namely the first case of (5.21).

Theorem 5.6 (Local Serrin condition, subcritical case)**.** *Let u be a weak solution to the surface growth model on a cylinder Q and let $q, q' \in (1, \infty]$ satisfy $1/q + 4/q' < 1$. If*

$$u_x \in L^{q',q}(Q)$$

then $u \in C^\infty(Q)$.

Proof. First, by translation we can assume that Q is contained within a time interval $(0, T)$ for some $T > 0$; that is $Q \Subset \mathbb{R} \times (0, T)$.

Secondly we can assume that $q' < \infty$. Indeed, since the choice of the exponents q', q is subcritical, the case $q' = \infty$ can be reduced to $q' < \infty$ by a use of Hölder's inequality,

$$\|u_x\|_{L^{q',q}(Q)} \le C\|u_x\|_{L^{\infty,q}(Q)},$$

where $q' < \infty$ is sufficiently large such that $1/q + 4/q' < 1$ holds.

Thirdly it suffices to prove the result under a smallness condition

$$\|u_x\|_{L^{q',q}(Q)} \le \delta \tag{5.25}$$

for $\delta > 0$ sufficiently small such that

$$\delta < \frac{1}{2} \max \left(C_{q,q',\infty,\infty}, C_{q,q',q/2,q'/2} \right)^{-1}, \tag{5.26}$$

where the constants on the right-hand side are from Theorem 5.1. Indeed, since $q' < \infty$, $\|u_x\|_{L^{q',q}(\widetilde{Q})} < \delta$ for every sufficiently small subcylinder $\widetilde{Q}$ of Q. Thus if $u \in C^\infty(\widetilde{Q})$ for every such cylinder, then the same is true for Q.

The proof of the result proceeds in a few steps.

Step 1. Show that $u_x \in L^\infty(\widetilde{Q})$ for any $\widetilde{Q} \Subset Q$.

Let $v := u_x$ and let $\phi \in C_0^\infty(Q; [0, 1])$ be such that $\phi = 1$ on $\widetilde{Q}$. By the representation (5.24) Corollary 5.2 applied with $r = r' = \infty$ gives

$$\|w\|_{\infty,\infty} \le C_{q,q',\infty,\infty}\|wv\|_{q',q} + C \left(\|v^2\|_{L^{q'/2,q/2}(Q)} + \|v\|_{L^{q',q}(Q)} \right) \tag{5.27}$$

Here we took $k = 3$, $l = q$, $l' = q'$ for the main nonlinearity (i.e. the term $-(wv)_{xxx}$ in (5.22)) as well as $l = q/2$, $l' = q'/2$ for the quadratic terms (i.e. the terms involving v^2

in (5.23)) and $l = q$, $l' = q'$ for the linear terms (i.e. the terms involving v in (5.23)). Note that here we have implicitly assumed that $q \geq 2$ (while $q' \geq 2$ follows from the assumption); the case $q \in (1,2)$ can be reduced to the case $q \geq 2$ by exploiting (5.20), which we explain in detail in Section 5.2.3 below.

Applying Hölder's inequality to the first term on the right-hand side of (5.27) gives

$$\|wv\|_{q',q} \leq \|w\|_{\infty,\infty} \|v\|_{L^{q',q}(Q)}$$

Thus given the smallness condition (5.26) we can absorb this term on the left hand side to see that

$$w \in L^\infty(Q) \tag{5.28}$$

and so in particular $v \in L^\infty(\widetilde{Q})$. Note however, that there is a gap in this step, since subtracting $\|w\|_{L^\infty(Q)}$ we have implicitly assumed that this norm is finite. If it is not the case then such argument may have a potentially fatal flaw.

This gap can be dealt with by a rather technical procedure of regularising the equation (5.22) and taking a limit of the solutions to the regularised equations, which we explain in more detail in the next step.

Step 1'. Verify (5.28)

Extend v by zero outside Q. Let $d \coloneqq \inf\{t \,:\, (x,t) \in Q\} > 0$. For every $\varepsilon \in (0, d/2)$ let $w^\varepsilon \in L^{\infty,\infty}$ be a solution of the problem

$$\begin{cases} w^\varepsilon_t + w^\varepsilon_{xxxx} = -\partial_{xxx}(v_\varepsilon w^\varepsilon) + f_{v_\varepsilon} & \text{in } \mathbb{R} \times (0,T), \\ w^\varepsilon(0) = 0, \end{cases} \tag{5.29}$$

where v_ε denotes the mollification of v (in both space and time), and the initial condition is understood in the sense that $w = 0$ in $\mathbb{R} \times (0,c)$ for some $c > 0$ independent of ε (one can take for example $c \coloneqq d$). The existence of such a w^ε follows from the Picard iteration, which we now briefly outline. Let

$$w^\varepsilon_0(t) \coloneqq \int_0^t \Phi(t-s) * f_{v_\varepsilon}(s)\,\mathrm{d}s, \qquad \text{for } t > 0,$$

and then set

$$w^\varepsilon_{m+1}(t) \coloneqq \int_0^t \Phi_{xxx}(t-s) * [v_\varepsilon(s) w^\varepsilon_m(s)]\,\mathrm{d}s + w^\varepsilon_0(t), \qquad \text{for } t > 0, m = 0, 1, \dots$$

Since $v_\varepsilon, f_{v_\epsilon} \in C_0^\infty(\mathbb{R}\times(0,T))$, Corollary 5.2 gives that $w_m^\varepsilon \in L^{\infty,\infty}$ for each m. Moreover each w_m^ε satisfies the equation

$$\partial_t w_m^\varepsilon + \partial_{xxxx} w_m^\varepsilon = -\partial_{xxx}(v_\varepsilon w_{m-1}^\varepsilon) + f_{v_\varepsilon} \quad \text{in } \mathbb{R}\times(0,T) \tag{5.30}$$

and $w_m^\varepsilon(t)=0$ for $t<d/2$. By Corollary 5.2

$$\begin{aligned}\|w_{m+1}^\varepsilon - w_m^\varepsilon\|_{\infty,\infty} &\le C_{q,q',\infty,\infty}\|v_\varepsilon(w_m^\varepsilon - w_{m-1}^\varepsilon)\|_{q',q} \le C_{q,q',\infty,\infty}\|v_\varepsilon\|_{q',q}\|w_m^\varepsilon - w_{m-1}^\varepsilon\|_{\infty,\infty}\\ &\le C_{q,q',\infty,\infty}\|v\|_{q',q}\|w_m^\varepsilon - w_{m-1}^\varepsilon\|_{\infty,\infty} \le \frac{1}{2}\|w_m^\varepsilon - w_{m-1}^\varepsilon\|_{\infty,\infty},\end{aligned}$$

where also used Hölder's inequality, the fact that mollification does not increase L^p norms and the smallness assumption (5.25). Thus $\{w_m^\varepsilon\}$ is a Cauchy sequence in $L^{\infty,\infty}$ and so

$$w_m^\varepsilon \to w^\varepsilon \quad \text{in } L^{\infty,\infty} \text{ as } m\to\infty$$

for some $w^\varepsilon \in L^{\infty,\infty}$ such that $w^\varepsilon(t)=0$ for $t<d/2$. Taking the limit $m\to\infty$ in (5.30) gives (5.29) (recall we mean partial differential equations in the distributional sense), which concludes the proof of the existence of w^ε.

We conclude this step by showing that $w^\varepsilon \overset{*}{\rightharpoonup} w$ as $\varepsilon\to 0$ (on some subsequence) in $L^{\infty,\infty}$ (and so in particular $w\in L^{\infty,\infty}$, as required).

Note that w^ε satisfies the representation

$$w^\varepsilon(t) := -\int_0^t \Phi_{xxx}(t-s) * [v_\varepsilon(s) w^\varepsilon(s)]\mathrm{d}s + w_0^\varepsilon(t)$$

(cf. (5.24)). Thus Corollary 5.2 gives

$$\begin{aligned}\|w^\varepsilon\|_{\infty,\infty} &\le C_{q,q',\infty,\infty}\|v_\varepsilon\|_{q',q}\|w^\varepsilon\|_{\infty,\infty} + C\left(\|v_\varepsilon^2\|_{q'/2,q/2} + \|v_\varepsilon\|_{q',q}\right)\\ &\le C_{q,q',\infty,\infty}\|v\|_{q',q}\|w^\varepsilon\|_{\infty,\infty} + C\left(\|v\|_{q',q}^2 + \|v\|_{q',q}\right),\end{aligned}$$

as in (5.27). Therefore the smallness condition (5.25) lets us absorb the first term on the right hand side to obtain

$$\|w^\varepsilon\|_{\infty,\infty} \le C$$

for all $\varepsilon>0$. In the same way one obtains

$$\|w^\varepsilon\|_{q',q} \le C$$

Hence there exists a sequence $\varepsilon_k\to 0$ and $\widetilde{w} \in L^{\infty,\infty}\cap L^{q',q}$ such that $w^{\varepsilon_k} \overset{*}{\rightharpoonup} \widetilde{w}$ in $L^{\infty,\infty}$ and $w^{\varepsilon_k} \rightharpoonup \widetilde{w}$ in $L^{q',q}$. Since $v\in L^{2,2}$ we see that $v_{\varepsilon_k}\to v$ in $L^{2,2}$ (as a property of the mollification operation) and thus we can take the limit in the partial differential equation in (5.29) (in the sense of distributions) to obtain that $\widetilde{w}$ and w satisfy the same partial differential equation and are both elements of $L^{q',q}$. It remains to show that $w = \widetilde{w}$. Let $h := w - \widetilde{w}$. Then $h \in L^{q',q}$, $h = 0$ in $\mathbb{R} \times (0, c)$ for some $c > 0$ (since the same is true for both w, $\widetilde{w}$) and

$$h_t + h_{xxxx} = -\partial_{xxx}(vh).$$

As in (5.24), h satisfies the representation formula,

$$h(t) = -\int_0^t \Phi_{xxx}(t-s) * [v(s)h(s)]\mathrm{d}s.$$

Thus Corollary 5.2 gives

$$\|h\|_{q',q} \le C_{q,q',q/2,q'/2}\|v\, h\|_{q'/2,q/2} \le C_{q,q',q/2,q'/2}\|v\|_{q',q}\|h\|_{q',q} \le \frac{1}{2}\|h\|_{q',q},$$

where we assumed that δ from the smallness condition (5.25) satisfies $\delta < C_{q,q',q/2,q'/2}/2$. Thus, since $\|h\|_{q',q} < \infty$, the above inequality implies $h = 0$, as required.

Step 2. Show boundedness of higher derivatives in x.

We proceed by induction. We will set $v^{(k)} := \partial_x^k v$ for brevity. We will show that if $v^{(k)} \in L^{\infty,\infty}(Q)$ then $v^{(k+1)} \in L^{\infty,\infty}(\widetilde{Q})$ for any subcylinder $\widetilde{Q} \Subset Q$.

Let $M > 1$ be such that $\|v\|_{L^{\infty,\infty}(Q)}, \ldots, \|v^{(k)}\|_{L^{\infty,\infty}(Q)} < M$ and let $\phi \in C_0^\infty(Q'; [0,1])$ be such that $\phi = 1$ on Q'' for some cylinders Q', Q'' such that $\widetilde{Q} \Subset Q'' \Subset Q' \Subset Q$. Let

$$z := v^{(k+1)}\phi.$$

Then z satisfies

$$\begin{aligned}
z_t + z_{xxxx} &= -(\partial_x^{k+4} v^2)\phi + 4v^{(k+4)}\phi_x + f_1 \\
&= -(\partial_x^{k+4} v^2)\phi + 4(v^{(k+1)}\phi_x)_{xxx} + f_1 + f_2 \\
&= -2(v^{(k+1)}v)_{xxx}\,\phi + 4(v^{(k+1)}\phi_x)_{xxx} + f_1 + f_2 + f_3 \\
&= -2(v\, z)_{xxx} + 4(v^{(k+1)}\phi_x)_{xxx} + f_1 + f_2 + f_3 + f_4
\end{aligned}$$

in $\mathbb{R} \times (0, T)$ (in the sense of distributions), where

$$\begin{cases}
f_1 := v^{(k+1)}\phi_t + 6v^{(k+3)}\phi_{xx} + 4v^{(k+2)}\phi_{xxx} + v^{(k+1)}\phi_{xxxx}, \\
f_2 := -12v^{(k+3)}\phi_{xx} - 12v^{(k+2)}\phi_{xxx} - 4v^{(k+1)}\phi_{xxxx}, \\
f_3 := -\sum_{j=1}^k \binom{k+1}{j} \left(v^{(j)}v^{(k+1-j)}\right)_{xxx}\phi, \\
f_4 := 6(v^{(k+1)}v)_{xx}\phi_x + 6(v^{(k+1)}v)_x\phi_{xx} + 2v^{(k+1)}v\,\phi_{xxx}.
\end{cases}$$

In short,

$$z_t + z_{xxxx} = -2(v\,z)_{xxx} + 4(v^{(k+1)}\phi_x)_{xxx} + F_{k+1}, \tag{5.31}$$

where

$$F_{k+1} \coloneqq f_1 + f_2 + f_3 + f_4$$

consists of (at most) second order derivatives (with respect to x) of (linear or quadratic) terms which include $v, \dots, v^{(k+1)}$ (and a derivative of the cutoff function ϕ).

Suppose for the moment that

$$v^{(k+1)} \in L^{2,2}(Q'). \tag{5.32}$$

Then $z \in L^{2,2}$ and $vz, v^{(k+1)}\phi_x \in L^{2,2}$ and so Corollary 5.3 gives the representation formula for z,

$$\begin{aligned} z(t) =& 2\int_0^t \Phi_{xxx}(t-s) * [v(s)z(s)]\mathrm{d}s - 4\int_0^t \Phi_{xxx}(t-s) * [v^{(k+1)}(s)\phi_x(s)]\mathrm{d}s \\ &+ \int_0^t \Phi(t-s) * F_{k+1}(s)\mathrm{d}s, \qquad t \in (0,T), \end{aligned} \tag{5.33}$$

where the last term is understood in the same sense as (5.14). Recalling that $z = v^{(k+1)}$ in Q'' and using Corollary 5.2 gives

$$\begin{aligned} \|v^{(k+1)}\|_{L^{r',r}(Q'')} \le \|z\|_{r',r} &\le C_\phi(\|vz\|_{l',l} + \|v^{(k+1)}\phi_x\|_{l',l} + M^2) \\ &\le C_\phi M(\|v^{(k+1)}\|_{L^{l',l}(Q')} + M^2) \end{aligned} \tag{5.34}$$

whenever $r, r', l, l' \in [1,\infty]$ satisfy

$$\frac{1}{l} + \frac{4}{l'} < \frac{1}{r} + \frac{4}{r'} + 1.$$

In other words we have obtained an increase in the integrability of $v^{(k+1)}$ with the cost of shrinking the domain slightly. It remains to bootstrap the inequality in (5.34). Namely let Q''' be a cylinder such that

$$\widetilde{Q} \Subset Q''' \Subset Q'' \Subset Q'$$

and let $\phi, \phi', \phi'' \in C_0^\infty(Q'; [0,1])$ be the cutoff functions such that ϕ' cuts-off Q''' in Q'' (i.e. $\phi' = 1$ on Q''' and $\phi' = 0$ outside Q'') and ϕ'' cuts-off $\widetilde{Q}$ in Q'''. Then apply (5.34) with $l' = l = 2$, $r' = r = 3$ to obtain

$$\|v^{(k+1)}\|_{L^{3,3}(Q'')} \le C_\phi M(\|v^{(k+1)}\|_{L^{2,2}(Q')} + M^2).$$

Taking $l' = l = 3$, $r' = r = 7$ we obtain

$$\|v^{(k+1)}\|_{L^{7,7}(Q''')} \le C_{\phi'} M(\|v^{(k+1)}\|_{L^{3,3}(Q'')} + M^2).$$

Finally the choice $l' = l = 7$, $r' = r = \infty$ gives

$$\|v^{(k+1)}\|_{L^{\infty,\infty}(\widetilde{Q})} \le C_{\phi''} M(\|v^{(k+1)}\|_{L^{7,7}(Q''')} + M^2),$$

as required. Therefore, in order to complete this step, it remains to verify (5.32).

Step 2'. Verify (5.32).

The case $k = 0$ follows from the definition of a weak solution (see Definition 5.5; recall also that $v = u_x$). In the case $k \ge 1$ let $\phi \in C_0^\infty(Q;[0,1])$ be such that $\phi = 1$ on a cylinder $\mathcal{Q}$ such that $Q' \Subset \mathcal{Q} \Subset Q$ and set

$$\eta \coloneqq v^{(k)}\phi.$$

As in (5.31), η satisfies

$$\eta_t + \eta_{xxxx} = -2(v\,\eta)_{xxx} + 4(v^{(k)}\phi_x)_{xxx} + F_k, \tag{5.35}$$

where F_k consists of (at most) second order derivatives (with respect to x) of terms consisting of bounded functions $v, \ldots, v^{(k)}$ (multiplied by a derivative of the cutoff function ϕ). Namely

$$F_k = \phi_k^{(2)} \partial_{xx} F_k^{(2)} + \phi_k^{(1)} \partial_x F_k^{(1)} + \phi_k^{(0)} F_k^{(0)},$$

where $F_k^{(0)}, F_k^{(1)}, F_k^{(2)} \in L^{\infty,\infty}(Q)$ and $\phi_k^{(l)} \in C_0^\infty(Q)$, $l = 0,1,2$. Now observe that we can absorb the functions $\phi_k^{(l)}$ into $F_k^{(l)}$, $l = 0,1,2$, by the chain rule. Namely there exist $G_k^{(0)}, G_k^{(1)}, G_k^{(2)} \in L^{\infty,\infty}(Q)$ such that

$$F_k = \partial_{xx} G_k^{(2)} + \partial_x G_k^{(1)} + G_k^{(0)}.$$

As in (5.33), η satisfies the representation

$$\begin{aligned} \eta(t) = 2\int_0^t \Phi_{xxx}(t-s) * [v(s)\eta(s)]\mathrm{d}s - 4\int_0^t \Phi_{xxx}(t-s) * [v^{(k)}(s)\phi_x(s)]\mathrm{d}s \\ + \int_0^t \Phi(t-s) * F_k(s)\mathrm{d}s, \qquad t \in (0,T). \end{aligned} \tag{5.36}$$

At this point we pause for a moment and comment on our strategy in an informal way. Formally, one could take the x-derivative in (5.36) to obtain

$$\begin{aligned}\eta_x(t) = 2 &\int_0^t \Phi_{xxx}(t-s) * [v_x(s)\eta(s) + v(s)\eta_x(s)]\mathrm{d}s \\ &- 4 \int_0^t \Phi_{xxx}(t-s) * [v^{(k+1)}(s)\phi_x(s) + v^{(k)}(s)\phi_{xx}(s)]\mathrm{d}s \\ &+ \int_0^t \Phi_x(t-s) * F_k(s)\mathrm{d}s, \qquad t \in (0,T).\end{aligned}$$

We would now like to use the estimates from Corollary 5.2 to obtain an estimate on $\|\eta_x\|_{2,2}$ and so deduce that $v^{(k+1)} \in L^{2,2}(\mathcal{Q})$. In fact, the terms including $v_x\eta$, $v^{(k)}\phi_{xx}$ and F_k could be dealt with easily, since $v,\ldots,v^{(k)} \in L^{\infty,\infty}(Q)$ (and $k \geq 1$), and the term including $v\eta_x$ could be dealt with by using the smallness condition (5.25) as in step 1'. However, the term including $v^{(k+1)}\phi_x$ (that is the one originating from the linear part, cf. the definition of f_1) cannot be dealt with in this way (as at this point we know nothing about $v^{(k+1)}$).

The way to deal with this problem is to increase the regularity of η by a "half of the derivative in x" at a time. Namely we will first show that $\Lambda^{1/2}\eta \in L^{2,2}$ and then deduce that $\Lambda\eta' \in L^{2,2}$, where

$$\eta' := v^{(k)}\phi' \tag{5.37}$$

and $\phi' \in C_0^\infty(\mathcal{Q};[0,1])$ is such that $\phi' = 1$ on Q'. Thus, since $2\pi\|\Lambda\eta'\|_{2,2} = \|\partial_x\eta'\|_{2,2}$ (recall (5.4)) we will obtain $\partial_x\eta' \in L^{2,2}$, and so in particular $v^{(k+1)} \in L^{2,2}(Q')$.

Taking the Fourier transform (in x) of (5.36) we obtain

$$\begin{aligned}\widehat{\eta}(t) = &- 16\pi^3 i \int_0^t \xi^3\widehat{\Phi}(t-s)\widehat{v\eta}(s)\mathrm{d}s + 32\pi^3 i \int_0^t \xi^3\widehat{\Phi}(t-s)\widehat{v^{(k)}\phi_x}(s)\mathrm{d}s \\ &+ \sum_{m=0}^{2}(2\pi i)^m \int_0^t \xi^m\widehat{\Phi}(t-s)\widehat{G_k^{(m)}}(s)\mathrm{d}s \qquad t \in (0,T).\end{aligned} \tag{5.38}$$

Multiplying (5.38) by $|\xi|^s$, where $s = s_1 + s_2$, $s_1, s_2 \in [0,1)$, taking the L^2 norm (in ξ) and using Plancherel's property we obtain

$$\begin{aligned}\|\Lambda^s\eta(t)\|_2 \leq &C \int_0^t \|\xi^{3+s_1}\widehat{\Phi}(\xi,t-s)\|_\infty \|\xi^{s_2}\widehat{v\eta}(\xi,s)\|_2\mathrm{d}s \\ &+ C \int_0^t \|\xi^{3+s_1}\widehat{\Phi}(\xi,t-s)\|_\infty \|\xi^{s_2}\widehat{v^{(k)}\phi_x}(\xi,s)\|_2\mathrm{d}s \\ &+ C \sum_{m=0}^{2} \int_0^t \|\xi^{m+s}\widehat{\Phi}(\xi,t-s)\|_\infty \|G_k^{(m)}(s)\|_2\mathrm{d}s \\ \leq &C \int_0^t \frac{\|\Lambda^{s_2}(v(s)\eta(s))\|_2 + \|\Lambda^{s_2}(v^{(k)}(s)\phi_x(s))\|_2}{(t-s)^{(3+s_1)/4}}\mathrm{d}s + C_T \int_0^t \frac{\|G_k(s)\|_2}{(t-s)^{(2+s)/4}}\mathrm{d}s,\end{aligned}$$

where $G_k = |G_k^{(0)}| + |G_k^{(1)}| + |G_k^{(2)}|$ and we also used (5.9) as well as applied the integral version of the Minkowski inequality. Taking the L^2 norm in time and using Young's inequality for convolutions we obtain

$$\|\Lambda^s \eta\|_{2,2} \le C_{s_1,s_2,\phi} \left(\|\Lambda^{s_2}(v\eta)\|_{2,2} + \|\Lambda^{s_2}(v^{(k)}\phi_x)\|_{2,2} + \|G_k\|_{2,2} \right) \tag{5.39}$$

for $s_1, s_2 \in [0,1)$, $s = s_1 + s_2$. Taking $s_1 = 1/2$, $s_2 = 0$ we see that $\Lambda^{1/2}\eta \in L^{2,2}$. Thus $\eta \in L^2((0,T); H^{1/2})$ and, thanks to the Sobolev–Slobodeckij characterisation (5.5),

$$\eta \in L^2((0,T); W^{1/2,2}(\mathbb{R})),$$

that is

$$\int_0^T \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|\eta(x,t) - \eta(y,t)|^2}{|x-y|^2} \mathrm{d}y \, \mathrm{d}x \, \mathrm{d}t < \infty.$$

Restricting the time domain to $\mathcal{I}$ and spatial domain to $\mathcal{B}$, where $\mathcal{Q} = \mathcal{B} \times \mathcal{I}$, we obtain that

$$\int_{\mathcal{I}} \int_{\mathcal{B}} \int_{\mathcal{B}} \frac{|v^{(k)}(x,t) - v^{(k)}(y,t)|^2}{|x-y|^2} \mathrm{d}y \, \mathrm{d}x \, \mathrm{d}t < \infty. \tag{5.40}$$

Thus

$$v^{(k)} \in L^2(\mathcal{I}; W^{1/2,2}(\mathcal{B}))$$

Now letting η' be the cutoff of $v^{(k)}$ as in (5.37) we can apply the triangle inequality and (5.40) to see that both $v\eta'$ and $v^{(k)}\phi'$ belong to $L^2(\mathcal{I}; W^{1/2,2}(\mathcal{B}))$ as well (recall that $v_x \in L^{\infty,\infty}$, since $k \ge 1$). Thus, since $v\eta'$ and $v^{(k)}\phi'$ are supported within $\mathcal{Q}$, they belong to $L^2((0,T); W^{1/2,2}(\mathbb{R})) = L^2((0,T); H^{1/2})$ (by the Sobolev–Slobodeckij characterisation (5.5)), and so

$$\|\Lambda^{1/2}(v\eta')\|_{2,2}, \|\Lambda^{1/2}(v^{(k)}\phi_x')\|_{2,2} < \infty.$$

Therefore, we can use (5.39) (applied to η', rather than η) with $s_1 = s_2 = 1/2$ to obtain

$$\|\eta_x'\|_{2,2} = 2\pi \|\Lambda \eta'\|_{2,2} < \infty,$$

where we have also recalled (5.4). In particular $\|v^{(k+1)}\|_{L^{2,2}(Q')} = \|\eta_x'\|_{L^{2,2}(Q')} < \infty$, as required.

Step 3. Deduce the smoothness of u.

From the surface growth equation, $u_t + u_{xxxx} + \partial_{xx} u_x^2 = 0$, and steps 1 and 2 we see that u_t (in the sense of weak derivatives) is bounded on every compact subset of Q.

Similarly every derivative (in both x and t) is bounded on every compact subset of Q. Thus the Rellich–Kondrachov embedding (see, for example, Theorem 6.3 in Adams & Fournier (2003)) gives smoothness of u in Q. □

5.2.2 The critical case

We now focus on the critical case, namely the second case of (5.21).

Theorem 5.7 (Local Serrin condition, critical case)**.** *Let u be a weak solution to the surface growth model on a cylinder Q and let $q, q' \in (1,\infty]$ satisfy $1/q + 4/q' = 1$. If*

$$u_x \in L^{q',q}(Q)$$

then $u \in C^\infty(Q)$.

Proof. Similarly as in Theorem 5.6 we can assume that $Q \Subset \mathbb{R} \times (0,T)$ and we can assume the smallness condition (5.25), since $q' < \infty$.

Let $\widetilde{Q} \Subset Q$ and fix $p' \in (q',\infty)$, $p \in (q,\infty)$ satisfying $1/p + 4/p' < 1$. We will show that $u_x \in L^{p',p}(\widetilde{Q})$ (then the claim follows from Theorem 5.6).

As in the proof of Theorem 5.6 let $v \coloneqq u_x$, $\phi \in C_0^\infty(Q;[0,1])$ be such that $\phi = 1$ on $\widetilde{Q}$. As in (5.24) $w \coloneqq v\phi$ satisfies the representation (5.24),

$$w(t) = -\int_0^t \Phi_{xxx}(t-s) * [w(s)v(s)] \,\mathrm{d}s + \int_0^t \Phi(t-s) * f_v(s) \,\mathrm{d}s. \tag{5.41}$$

Corollary 5.2 applied with $r = p$, $r' = p'$ gives

$$\|w\|_{p',p} \le C_{l,l',p,p'} \|wv\|_{l',l} + C\left(\|v^2\|_{L^{q'/2,q/2}(Q)} + \|v\|_{L^{q',q}(Q)} \right), \tag{5.42}$$

where $1/l = 1/p + 1/q$, $1/l' = 1/p' + 1/q'$ (note $l \in (q,p)$, $l' \in (q',p')$), cf. (5.27). (Note that again we have implicitly assumed that $q > 2$, see the section below for the case $q \in (1,2]$.) Applying Hölder's inequality to first term on the right-hand side of (5.42) and using the smallness condition (5.25) with $\delta < 1/2C_{l,l',p,p'}$ gives

$$\|w\|_{p',p} \le \frac{1}{2}\|w\|_{p',p} + C\left(\|v\|^2_{L^{q',q}(Q)} + \|v\|_{L^{q',q}(Q)} \right) \tag{5.43}$$

Thus, subtracting the first term on the right hand side we obtain $w \in L^{p',p}(Q)$, which gives in particular that $v \in L^{p',p}(\widetilde{Q})$, as required.

Note that, similarly as in step 1 of the proof of Theorem 5.6, this subtraction requires a rigorous justification, and can be verified similarly as in step 1' of that proof. □

5.2.3 The case $q \in (1, 2]$

Here we briefly show that if q, q' satisfy $q \in (1, 2]$ and $1/q + 4/q' \leq 1$ then one can apply a similar argument as in (5.42) to obtain that $v \in L^{p',p}(Q')$ for $p = 2$, any $p' \in (\max(q', 8/3), \infty)$ and any subcylinder $Q' \Subset Q$. Note that, since $1/p + 4/p' < 1$, the claim (i.e. $u_x \in C^\infty(Q)$) then follows by Theorem 5.6.

In order to show that $v \in L^{p',p}(Q')$ observe that the last term of (5.41) can be bounded in the $L^{p',p}$ norm by

$$C \left(\|v^2\|_{L^{8/3,1}(Q)} + \|v\|_{L^{q',q}(Q)} \right),$$

where we used Corollary 5.2 again (recall that $v \in L^{16/3,2}(Q)$ (see (5.20)) and observe that the choice of exponents $r' = p'$, $r = 2$, $l' = 8/3$, $l = 1$ satisfies (5.11) with $k = 2$). Thus (5.42) follows as in the proof above with $\|v^2\|_{L^{q'/2,q/2}(Q)}$ replaced by $\|v^2\|_{L^{8/3,1}(Q)}$. As in (5.43) we arrive at

$$\|w\|_{p',p} \leq \frac{1}{2}\|w\|_{p',p} + C \left(\|v\|^2_{L^{16/3,2}(Q)} + \|v\|_{L^{q',q}(Q)} \right),$$

from which we deduce that $v \in L^{p',p}(\widetilde{Q})$ for any $\widetilde{Q} \Subset Q$ (as in the proof above; namely by regularising the equation and taking a limit as in step 1' of the proof of Theorem 5.6).

Finally, observe that the issue discussed in this section does not appear in the case of the Navier–Stokes equation. In fact, in the case of the NSE the dimension of space is larger that the order of the nonlinearity (i.e. 2); to be more precise the constraint of the exponents $2/q' + 3/q \leq 1$ (recall (5.1)) implies $q \geq 3 > 2$.

Chapter 6

Conclusion

We have presented examples that show the sharpness of the bound $d_H(S) \leq 1$ for weak solutions of the Navier–Stokes inequality. While it is still not known whether the result of Caffarelli et al. (1982) can be improved for suitable weak solutions, the examples show that it cannot be improved for weak solutions of the Navier–Stokes inequality.

This work leads to several open problems.

Q1. Is it possible to construct a weak solution to the NSI that blows up on a set of dimension 1 (rather than of dimension that is arbitrarily close to 1)?

Q2. Can one construct a weak solution to the NSI that blows up at a number of times, possibly with an accumulation point? Such solutions could provide an example of the singular set S with $d_H(S) \leq 1$, but $d_B(S) > 1$ (recall that $d_H(C) = 0$ for any countable set C, which is not the case for d_B), and thus give new insight into the study of the structure of the singular set. It would also help understand the difference between the solutions of the NSE and the solutions of the NSI.

Q3. Is there any relation between $d_H(S)$ and $d_B(S)$ in the case of (suitable) weak solutions of the NSE? Is it possible, perhaps, to show that $d_H(S) = d_B(S)$?

Q4. Does the (putative) singular set S admit any particular structure (or shape)? For example, is it a Cantor set (as in Chapter 2), or does it satisfy any other self-similar property? This would limit the possible complexity of the structure of space-time singularities, should they exist.

It is also interesting to consider the singular set in space for the first blow-up time. In this context Scheffer (1976a) showed that the 1-dimensional Hausdorff measure of the singular set (in space) at the first blow-up time is finite. Later Seregin (2001) estimated the number of singular points of a suitable weak solution u at a given time in terms of the local behaviour of the quantity $|u|^3$. Moreover, Katz & Pavlović (2002) use the Littlewood-Paley theory to show that if the Laplacian term "$-\nu\Delta u$" in the Navier–Stokes eqations (1.1) is replaced by (appropriately defined) fractional Laplacian "$\nu(-\Delta)^\alpha u$", with $\alpha \in (1, 5/4)$, then the Hausdorff dimension of the singular set at the first blow-up time is bounded by $5\alpha - 4$. Therefore, since such modified equations are regular (that is no blow-up occurs) for $\alpha > 5/4$ (see, for example, Proposition 1.1 in Katz & Pavlović (2002)), this result provides, in a sense, a "link" between the result of Scheffer (1976*a*) and the "full" regularity for the equations with stronger dissipation. It is not know whether such a result holds for the surface growth model, and this is work in progress.

In Chapter 4 we proved two partial regularity results for the surface growth model, $u_t + u_{xxxx} + \partial_{xx} u_x^2 = 0$. These results allow us to estimate the dimension of the singular set S,

$$d_H(S) \leq 1, \qquad d_B(S \cap K) \leq 7/6$$

for any compact set K, where S consists of points where u is not locally Hölder continuous. A natural question to ask is whether the bound on $d_B(S)$ can be improved, just as in the case of the NSE. Perhaps, again, $d_H(S) = d_B(S)$.

Moreover, does the partial regularity theory apply for solutions of the inequality

$$u \left(u_t + u_{xxxx} + \partial_{xx} u_x^2 \right) \leq 0?$$

This would be the case if one could adapt the iterative scheme from the original proof of the partial regularity for the NSE due to Caffarelli, Kohn & Nirenberg (1982) to the SGM. If so, then can one construct a solution of such inequality that blows up in finite time? In other words, do any ideas from Chapter 2 transfer to the SGM? This seems difficult, since the constructions make use of (i) the three-dimensional nature of the fluid flow and (ii) the pressure function plays a fundamental role in amplifying the magnitude of the velocity.

Furthermore, it is still not clear whether the definition of the singular set S is optimal, that is whether the complement of S (that is the set where u is locally Hölder continuous) consist of regular points. Thus, is it true that a suitable weak solution to the SGM, which is locally Hölder continuous, is smooth in space-time? This would be true if any of the local conditions

$$u \in L^\infty(Q), \qquad u \in L^{8/(2\alpha-1)}(I; H^\alpha(B))$$

guarantee smoothness on a given space-time cylinder $Q = I \times B$. Although this is an open question, the local regularity condition from Chapter 5 suggests that it might be true.

The results of Chapter 5 show that the analogue of the local Serrin condition for the Navier–Stokes equations hold for the SGM. That is, if a weak solution u to the SGM satisfies the additional integrability property $u_x \in L^{q'}(I; L^q(B))$ with $q \in [2, \infty]$, $q' \in [4, \infty]$ such that

$$\frac{1}{q} + \frac{4}{q'} < 1 \qquad \text{or} \qquad \frac{1}{q} + \frac{4}{q'} = 1, q' < \infty,$$

then u is smooth in $Q = I \times B$. Our analysis excludes the endpoint case $q = 1$, $q' = \infty$ since in this case we cannot assume the smallness condition (5.25). This case is not only particularly interesting, but also particularly challenging. In fact, the analogous problem in the case of the Navier–Stokes equations is the problem whether a weak solution $u \in L^\infty(I; L^3(B))$ to the Navier–Stokes equations on Q is regular in Q, and it was resolved in a deep paper by Escauriaza, Seregin & Šverák (2003) using a blow-up technique. We believe that, despite the fact that the surface growth model is a one-dimensional equation, it might be a more difficult problem than in the case of the Navier–Stokes equations, since a number of techniques used by Escauriaza et al. (2003) seem unavailable in the surface growth model (such as the $L_{s,l}$-coercive estimates for solutions of the non-stationary Stokes system) and since the L^1 space is not reflexive.

However, given the number of similarities between the surface growth model and the Navier–Stokes equations, as developed in the recent years, we conjecture that the case $q = 1$, $q' = \infty$ gives regularity as well.

One of the challenges of the Navier–Stokes regularity problem is to try to find a method that would quantify the "strength" of each of the diffusive term "Δu" (the "good term") and the nonlinear term "$(u \cdot \nabla)u$" (the "bad term") in the Navier–Stokes equations (1.1) in an effective way. It seems very hard, roughly speaking, to estimate how the energy of the solution is moved around by these terms. Moreover, the role of the pressure term "∇p" seems mysterious. In fact, the considerations of Chapters 2 and 3 show some geometric properties of the pressure function that enables one to engineer a solution of the Navier–Stokes equations with damping (that is with a force acting against the direction of the flow, recall (1.25)) in which the energy concentrates (around a point or a Cantor set) to create a blow-up. It therefore seems possible that is the pressure function that, roughly speaking, is an "invisible player" who is responsible for all the trouble.

Furthermore, it appears that studying the surface growth model could give some useful insight into studying such flow of energy. Indeed, due to the lack of the pressure function, it seems easier to study the interaction between the "good term" u_{xxxx} and the "bad term" $\partial_{xx} u_x^2$ in this one-dimensional scalar model. Therefore we think that it is potentially more feasible to understand the local behaviour of energy of solutions to the surface growth model. We believe that not only the regularity problem of the SGM should be easier to answer (than in the case of the NSE), but we also hope that the study of the SGM could provide us with new tools for studying the regularity problem of the NSE.

Bibliography

Adams, R. A. & Fournier, J. J. F. (2003), *Sobolev spaces*, Vol. 140 of *Pure and Applied Mathematics (Amsterdam)*, second edn, Elsevier/Academic Press, Amsterdam.

Aramaki, J. (2016), 'Poincaré inequality and Campanato estimates for weak solutions of parabolic equations', *Electronic J. Diff. Eq.* (204), 1–8.

Banach, S. (1938). Foreword to E. Colerus' *Od tabliczki do ró zniczki* (*From multiplication table to calculus*), Państwowe Wydawnictwo książek szkolnych, Lviv.

Beale, J. T., Kato, T. & Majda, A. (1984), 'Remarks on the breakdown of smooth solutions for the 3-D Euler equations', *Comm. Math. Phys.* **94**.

Biryuk, A., Craig, W. & Ibrahim, S. (2007), Construction of suitable weak solutions of the Navier-Stokes equations, *in* 'Stochastic analysis and partial differential equations', Vol. 429 of *Contemp. Math.*, Amer. Math. Soc., Providence, RI, pp. 1–18.

Blömker, D. & Romito, M. (2009), 'Regularity and blow up in a surface growth model.', *Dyn. Partial Differ. Equ.* **6**(3), 227–252.

Blömker, D. & Romito, M. (2012), 'Local existence and uniqueness in the largest critical space for a surface growth model.', *Nonlinear Differ. Equ. Appl.* **19**(3), 365–381.

Buckmaster, T. & Vicol, V. (2017), 'Nonuniqueness of weak solutions to the Navier–Stokes equation'. arXiv:1709.10033.

Caffarelli, L., Kohn, R. & Nirenberg, L. (1982), 'Partial regularity of suitable weak solutions of the Navier-Stokes equations', *Comm. Pure Appl. Math.* **35**(6), 771–831.

Constantin, P. & Fefferman, C. (1993), 'Direction of vorticity and the problem of global regularity for the Navier-Stokes equations', *Indiana Univ. Math. J.* **42**(3), 775–789.

Conte, S. D. & de Boor, C. (1972), *Elementary numerical analysis: An algorithmic approach*, McGraw-Hill Book Co., New York-Toronto, Ont.-London.

Di Nezza, E., Palatucci, G. & Valdinoci, E. (2012), 'Hitchhiker's guide to the fractional Sobolev spaces', *Bull. Sci. Math.* **136**(5), 521–573.

Escauriaza, L., Seregin, G. A. & Šverák, V. (2003), '$L_{3,\infty}$-solutions of Navier-Stokes equations and backward uniqueness', *Russian Math. Surveys* **58**(2), 211–250.

Fabes, E. B., Jones, B. F. & Rivière, N. M. (1972), 'The initial value problem for the Navier-Stokes equations with data in L^p', *Arch. Rational Mech. Anal.* **45**, 222–240.

Falconer, K. (2014), *Fractal geometry - Mathematical foundations and applications*, John Wiley & Sons, Ltd., Chichester. Third edition.

Ferrero, A., Gazzola, F. & Grunau, H.-C. (2008), 'Decay and eventual local positivity for biharmonic parabolic equations', *Discrete Contin. Dyn. Syst.* **21**(4), 1129–1157.

Giga, M.-H., Giga, Y. & Saal, J. (2010), *Nonlinear partial differential equations, Asymptotic behavior of solutions and self-similar solutions*, Progress in Nonlinear Differential Equations and their Applications, 79, Birkhäuser Boston, Inc., Boston, MA.

He, C., Wang, Y. & Zhou, D. (2017), 'New ε-regularity criteria and application to the box-counting dimension of the singular set in the 3D Navier–Stokes equations'. arXiv:1709.01382.

Hopf, E. (1951), 'Über die Anfangswertaufgabe für die hydrodynamischen Grundgleichungen', *Math. Nachr.* **4**, 213–231. English translation by Andreas Klöckner.

Katz, N. H. & Pavlović, N. (2002), 'A cheap Caffarelli-Kohn-Nirenberg inequality for the Navier-Stokes equation with hyper-dissipation', *Geom. Funct. Anal.* **12**(2), 355–379.

King, B. B., Stein, O. & Winkler, M. (2003), 'A fourth-order parabolic equation modeling epitaxial thin film growth', *J. Math. Anal. Appl.* **286**(2), 459–490.

Koch, H. & Tataru, D. (2001), 'Well-posedness for the Navier-Stokes equations', *Adv. Math.* **157**(1), 22–35.

Koh, Y. & Yang, M. (2016), 'The Minkowski dimension of interior singular points in the incompressible Navier-Stokes equations', *J. Differential Equations* **261**(6), 3137–3148.

Kukavica, I. (2009*a*), 'The fractal dimension of the singular set for solutions of the Navier-Stokes system', *Nonlinearity* **22**(12), 2889–2900.

Kukavica, I. (2009*b*), Partial regularity results for solutions of the Navier-Stokes system, *in* 'Partial differential equations and fluid mechanics', Vol. 364 of *London Math. Soc. Lecture Note Ser.*, Cambridge Univ. Press, Cambridge, pp. 121–145.

Kukavica, I. (2009*c*), Partial regularity results for solutions of the Navier-Stokes system, *in* 'Partial differential equations and fluid mechanics', Vol. 364 of *London Math. Soc. Lecture Note Ser.*, Cambridge Univ. Press, Cambridge, pp. 121–145.

Kukavica, I. & Pei, Y. (2012), 'An estimate on the parabolic fractal dimension of the singular set for solutions of the Navier-Stokes system', *Nonlinearity* **25**(9), 2775–2783.

Ladyzhenskaya, O. A. & Seregin, G. A. (1999), 'On partial regularity of suitable weak solutions to the three-dimensional Navier-Stokes equations', *J. Math. Fluid Mech.* **1**(4), 356–387.

Leray, J. (1934), 'Sur le mouvement d'un liquide visqueux emplissant l'espace', *Acta Math.* **63**, 193–248. (An English translation due to Robert Terrell is available at *http://www.math.cornell.edu/ bterrell/leray.pdf* and *https://arxiv.org/abs/1604.02484.*).

Lieberman, G. M. (2005), *Second Order Parabolic Differential Equations*, World Scientific.

Lin, F. (1998), 'A new proof of the Caffarelli-Kohn-Nirenberg theorem', *Comm. Pure Appl. Math.* **51**(3), 241–257.

McCormick, D. S., Robinson, J. C. & Rodrigo, J. L. (2013), 'Generalised Gagliardo-Nirenberg inequalities using weak Lebesgue spaces and BMO', *Milan J. Math.* **81**(2), 265–289.

Nečas, J., Růžička, M. & Šverák, V. (1996), 'On Leray's self-similar solutions of the Navier-Stokes equations', *Acta Math.* **176**(2), 283–294.

Ożański, W. S. (2017), 'On weak solutions to the Navier–Stokes inequality with internal singularities'. Submitted; preprint available at arXiv:1709.00602.

Ożański, W. S. (2018), 'A sufficient integral condition for local regularity of solutions to the surface growth model'. Submitted; preprint available at arXiv:1803.08913.

Ożański, W. S. & Pooley, B. C. (2017), 'Leray's fundamental work on the Navier-Stokes equations: a modern review of *"Sur le mouvement d'un liquide visqueux emplissant l'espace"*'. To appear in *Partial Differential Equations in Fluid Mechanics*, edited by C. Fefferman, J. C. Robinson, J. L. Rodrigo, LMS Lecture Notes Series, Cambridge University Press; preprint available at arXiv:1708.09787.

Ożański, W. S. & Robinson, J. C. (2017), 'Partial regularity for a surface growth model'. Submitted; preprint available at arXiv:1710.07625.

Raible, M., Linz, S. J. & Hänggi, P. (2000), 'Amorphous thin film growth: Minimal deposition equation', *Phys. Rev. E* **62**, 1691–1694.

Robinson, J. C., Rodrigo, J. L. & Sadowski, W. (2016), *The three-dimensional Navier-Stokes equations*, Vol. 157 of *Cambridge Studies in Advanced Mathematics*, Cambridge University Press, Cambridge.

Robinson, J. C. & Sadowski, W. (2007), 'Decay of weak solutions and the singular set of the three-dimensional Navier-Stokes equations', *Nonlinearity* **20**(5), 1185–1191.

Robinson, J. C. & Sadowski, W. (2009), 'Almost-everywhere uniqueness of Lagrangian trajectories for suitable weak solutions of the three-dimensional Navier-Stokes equations', *Nonlinearity* **22**(9), 2093–2099.

Robinson, J. C. & Sharples, N. (2013), 'Dimension prints and the avoidance of sets for flow solutions of non-autonomous ordinary differential equations', *J. Differential Equations* **254**(10), 4144–4167.

Scheffer, V. (1976*a*), 'Partial regularity of solutions to the Navier-Stokes equations', *Pacific J. Math.* **66**(2), 535–552.

Scheffer, V. (1976*b*), 'Turbulence and Hausdorff dimension'. In *Turbulence and Navier-Stokes equations* (Proc. Conf., Univ. Paris-Sud, Orsay, 1975), Springer LNM 565: 174–183, Springer Verlag, Berlin.

Scheffer, V. (1977), 'Hausdorff measure and the Navier-Stokes equations', *Comm. Math. Phys.* **55**(2), 97–112.

Scheffer, V. (1978), 'The Navier-Stokes equations in space dimension four', *Comm. Math. Phys.* **61**(1), 41–68.

Scheffer, V. (1980), 'The Navier-Stokes equations on a bounded domain', *Comm. Math. Phys.* **73**(1), 1–42.

Scheffer, V. (1985), 'A solution to the Navier-Stokes inequality with an internal singularity', *Comm. Math. Phys.* **101**(1), 47–85.

Scheffer, V. (1987), 'Nearly one-dimensional singularities of solutions to the Navier-Stokes inequality', *Comm. Math. Phys.* **110**(4), 525–551.

Seregin, G. (2007), 'Navier-Stokes equations: almost $L_{3,\infty}$-case', *J. Math. Fluid Mech.* **9**(1), 34–43.

Seregin, G. (2012), 'A certain necessary condition of potential blow up for Navier-Stokes equations', *Comm. Math. Phys.* **312**(3), 833–845.

Seregin, G. A. (2001), 'On the number of singular points of weak solutions to the Navier-Stokes equations', *Comm. Pure Appl. Math.* **54**(8), 1019–1028.

Serrin, J. (1962), 'On the interior regularity of weak solutions of the Navier-Stokes equations', *Arch. Rational Mech. Anal.* **9**, 187–195.

Serrin, J. (1963), The initial value problem for the Navier-Stokes equations, *in* 'Nonlinear Problems (Proc. Sympos., Madison, Wis., 1962)', Univ. of Wisconsin Press, Madison, Wis., pp. 69–98.

Siegert, M. & Plischke, M. (1994), 'Solid-on-solid models of molecular-beam epitaxy', *Physical Review E* **50**, 917–931.

Stein, O. & Winkler, M. (2005), 'Amorphous molecular beam epitaxy: global solutions and absorbing sets', *European J. Appl. Math.* **16**(6), 767–798.

Struwe, M. (1981), 'On the Hölder continuity of bounded weak solutions of quasilinear parabolic systems', *Manuscripta Math.* **35**(1-2), 125–145.

Struwe, M. (1988), 'On partial regularity results for the Navier-Stokes equations', *Comm. Pure Appl. Math.* **41**(4), 437–458.

Takahashi, S. (1990), 'On interior regularity criteria for weak solutions of the Navier-Stokes equations', *Manuscripta Math.* **69**(3), 237–254.

Temam, R. (2001), *Navier-Stokes equations*, AMS Chelsea Publishing, Providence, RI. Theory and numerical analysis, Reprint of the 1984 edition.

Vasseur, A. F. (2007), 'A new proof of partial regularity of solutions to Navier-Stokes equations', *Nonlinear Diff. Equ. Appl.* **14**(5-6), 753–785.